Thayer Prize Exam in Mathematics

for Dartmouth First-year students

Saturday April 30, 2022
10 AM - 1 PM
If there are any exam scheduling conflicts, you are welcome to take it on Sunday May 1

PRINT NAME: ______________________________

Acknowledgment: Some of the problems are inspired by problems in recent math competitions in the US and in Russia, and by problems from other sources.

Honor Code: You are not allowed to give or receive any help on this exam. Using calculators or computers is not allowed. You have 3 hours to work on the exam.

Grader's use only

1. ______________ /10
2. ______________ /10
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8. ______________ /10
9. ______________ /10

Total: __________ /90

(1) Let ABC be a triangle such that for every point P inside of it one can construct a triangle with the sides equal to the vectors PA, PB and PC. Show that the triangle ABC is equilateral.

(2) Let A, B be two square matrices such that $A + B = AB$. Prove that the matrices commute that is $AB = BA$.

(3) Let $S^1 = \{e^{i\theta} : \theta \in [0, 2\pi)\}$ be the unit circle in the complex plane, equipped with the arclength distance $d(e^{i\theta}, e^{i\phi}) = \min\{|\theta - \phi|, \pi - |\theta - \phi|\}$. Consider the map $f : S^1 \to S^1$ given by $f(e^{i\theta}) = e^{3i\theta}$. Fix an angle $\alpha \in (0, \pi/2)$. Show that for every point $p \in S^1$ and $\epsilon > 0$ there exists a point $q \in S^1$ with $d(p, q) < \epsilon$ such that $d(f^k(p), f^k(q)) \geq \alpha$ for some integer $k > 0$.

(4) Consider the initial-value problem $x'(t) = f(x(t))$, $x(0) = x_0$, where $f : \mathbb{R} \to \mathbb{R}$ is a continuous function satisfying $f(x_0) = 0$ for some point x_0. Show that a sufficient condition for uniqueness of the solution $x(t) = x_0$ is that $\lim_{\epsilon \to 0} \int_{x_0}^{x_0+\epsilon} \frac{dx}{f(x)}$ is infinite or does not exist.

(5) Find all integers n such that $(n^2 + 1)$ divides $(n^2 - 3n)$.

(6) Recall that a complex *root of unity* is a complex number $z \in \mathbb{C}$ such that there exists a positive integer $n \geq 1$ such that $z^n = 1$.

Find all complex roots of unity w, z such that

$$z^5 + w^5 = 1.$$

(7) You have a sack containing n objects, all distinct, and you define a game as follows. You draw one object uniformly at random and replace it. You draw a second time and replace it. You then draw a third item. You win the game if you happen to draw one of the n objects exactly two times; otherwise, you lose.

You play this game n times. Calulate the expected number of wins as $n \to \infty$.

(8) Consider the modified Harmonic series $\sum_{i=1}^{\infty} \frac{1}{i}$ with all the terms containing the digit 9 deleted. So that the terms $\frac{1}{9}$, $\frac{1}{19}$ $\frac{1}{90}$, $\frac{1}{91}$ are deleted. Prove or disprove that this modified series is convergent.

(9) Two marksmen, one of whom ("Acuron") hits a certain small target 75% of the time and the other ("Blunderon") only 25%, aim simultaneously at that target. One bullet hits. What's the probability that it came from Acuron?