Math 103: Metric spaces and measure theory

ORC syllabus

This course reviews the basic theory of metric spaces and their topology including continuity, completeness, connectedness, and compactness. An introduction to abstract measure theory follows, with topics including measurability, measures, integration, the construction of Lebesgue measure, as well as additional topics as time allows.

References.

[RF] H. L. Royden and P. M. Fitzpatrick, *Real analysis*, 4th ed., Prentice Hall, 2010.

[F] G. B. Folland, *Real analysis*, 2nd ed., Wiley, 2009.

[R] W. Rudin, *Real & complex analysis*, 3rd ed., McGraw-Hill, 1986.

Metric spaces. [RF, 9,10].

1. Metrics and norms.
2. Continuity and uniform continuity.
3. Completeness.
4. Compactness. Total boundedness and sequential compactness. Compactness in $\mathbb{R}^n$. Lebesgue numbers.
5. Arzelá–Ascoli theorem. Equicontinuity. Compact sets in $C(X)$.
6. Baire category theorem.
7. Banach contraction principle.
8. Completion of a metric space.

Abstract measure theory. [RF, 17.1–17.5, 18.1–18.4, 19.1, 20.1] and [F, 1.1–1.4, 2.1–2.5, 3.1–3.2].

1. Review of the Riemann integral. Shortcomings and the idea of the Lebesgue integral.
2. Measurable spaces. σ-algebras. The Borel σ-algebra in a topological space.
3. Measurable functions. Borel functions.

4. Integrals. Simple functions and integrals of nonnegative functions. Monotone convergence theorem. Fatou's lemma. Integrable functions and the dominated convergence theorem.

5. Lebesgue measure. Outer measures and basic Carathéodory. Lebesgue measure on $\mathbb{R}$. Lebesgue measurable sets and the Riemann integral. Cantor set.

6. Banach spaces. $L^1(X, \mathcal{M}, \mu)$ and convergence (via convergence in measure). Littlewood's three principles. Egoroff's theorem. Lusin's theorem. $C_c(\mathbb{R})$ is dense in $L^1(\mathbb{R})$.

7. Radon–Nikodym theorem. Real-valued measures. Hahn decomposition.

8. Product measures. Tonelli's theorem. Fubini's theorem. Completion.

9. L^p-spaces. Hölder inequality. Minkowski inequality. Completeness.