# Sample solutions to questions for algebra preliminary exam

**Problem 1**. Let $A$ be a real $2 \times 2$ matrix such that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector with eigenvalue 3 and such that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue $-2$. Compute $A^{-1}$ applied to $\begin{pmatrix} 7 \\ 8 \end{pmatrix}$.

*Solution.* One way to proceed is to determine $A$ via a "reverse diagonalization"

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}^{-1}$$

then calculate $A^{-1}$, then apply it to the required vector.

A more elegant way is to note that if $v$ is an eigenvector of $A$ with eigenvalue $\lambda$ then $v$ is an eigenvector of $A^{-1}$ with eigenvalue $\lambda^{-1}$. Then we simply decompose $\begin{pmatrix} 7 \\ 8 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2 \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and then apply

$$A^{-1} \begin{pmatrix} 7 \\ 8 \end{pmatrix} = 3A^{-1} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2A^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 3\frac{1}{3} \begin{pmatrix} 1 \\ 2 \end{pmatrix} + 2\frac{1}{-2} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

**Problem 2**. Determine whether the surface in $\mathbb{R}^3$ defined by

$$x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 1$$

is a ellipsoid (an ellipse rotated about one of its axes) or a hyperboloid (a hyperbola rotated about one of its axes).

*Solution.* Since the surface is defined by a quadratic equation, we can rewrite the equation as $X^t A X = 1$, where

$$A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

By the spectral theorem for symmetric operators on a real vector space, we can find an orthogonal matrix $Q$ (whose rows are an orthonormal basis of eigenvectors of $A$) and a diagonal matrix $D$ (whose diagonal elements are the eigenvalues) so that $A = QDQ^{-1}$. Since $Q^{-1} = Q^t$, the equation becomes $(Q^t X)^t D(Q^t X) = 1$, showing that $Q^t$ transforms the equation to "standard" diagonal form. Since the charateristic polynomial of $A$ is (up to a sign) $t^3 - 3t^2 + 4 = (t+1)(t-2)^2$, the eigenvalues of $A$ have sign $-, +, +$: since these are not all positive, the surface is a (one-sheeted) hyperboloid.

**Problem 3**. Let $n \in \mathbb{Z}_{\geq 1}$, and let $V$ be the vector space of real polynomials of degree at most $n$. Consider the linear operator $T \colon V \to V$ defined by $T(f)(x) = f(1-x)$.

(a) Compute the determinant of $T$.

(b) Consider the bilinear form on $V$ defined by $\langle f, g \rangle = \int_0^1 f(x)g(x)\,\mathrm{d}x$. Show that $T$ is self-adjoint with respect to this inner product.

(c) For $n = 2$, find a basis of $V$ consisting of eigenvectors for $T$.

*Solution.* For (a), the determinant of $T$ can be calculated on any basis, so we can use $1, x, x^2, \ldots, x^n$. Since $(1-x)^n = (-1)^n x^n + \cdots + (-1)^n$, the matrix of $T$ with respect to this basis is upper-triangular, with entries alternating in sign, e.g., for $n = 2$:

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & 0 & 1 \end{pmatrix}$$

and hence the determinant of $T$ is $(-1)^{\lceil n/2 \rceil} = 1, -1, -1, 1, 1, -1, -1, \ldots$.

For (b), we compute

$$\langle T(f), g \rangle = \int_0^1 f(1-x)g(x)\,dx = \int_0^1 f(u)g(1-u)\,du = \langle f, T(g) \rangle$$

by employing the $u$-substitution $u = 1 - x$. Hence $T$ is self-adjoint.

For (c), by looking at the matrix above, we know the eigenvalues of $T$. So we are looking to solve the equation $f(1-x) = \pm f(x)$. Clearly $f(x) = 1$ is an eigenvector with eigenvalue $\lambda = 1$; the other is $f(x) = x(1-x) = x - x^2$. Finally, $f(x) = x - (1-x) = -1 + 2x$ is an eigenvector with eigenvalue $-1$.

**Problem 4**. Let $V$ be a finite-dimensional vector space and $T\colon V \to V$ a linear operator satisfying $T^2 = T$.

(a) Show that the only possible eigenvalues of $T$ are zero or one.

(b) If $E_\lambda$ is the $\lambda$-eigenspace, show that $E_0 = \ker T$ and $E_1$ is the image of $T$.

(c) Show that $T$ is diagonalizable.

*Solution.* For (a), one could observe that the minimal polynomial must divide $x(x-1)$, or missing that try the direct approach. Suppose that $T(v) = \lambda v$ for some nonzero $v$. Then

$$T(v) = T^2(v) = T(T(v)) = T(\lambda v) = \lambda T(v),$$

so that $(\lambda - 1)T(v) = 0$, so either $\lambda = 1$, or $T$ has nontrivial kernel, so has eigenvalue 0.

For (b), by definition $E_0 = \ker T$. If $v \in E_1$, then $T(v) = v$, so $v$ is in the image. Conversely, if $w$ is in the image of $T$, say $w = T(v)$, then

$$T(w) = T^2(v) = T(v) = w,$$

so $w \in E_1$.

For (c), this follows from the fact that the minimal polynomial is separable (has no repeated roots), or directly: since $\dim E_0 = \operatorname{nullity} T$ and $\dim E_1 = \operatorname{rk} T$ and $E_1 \cap E_0 = \{0\}$, rank-nullity says that $V$ has a basis consisting of eigenvectors for $T$.

**Problem 5**. Let $V$ be a finite-dimensional vector space over a field $F$, let $V^* := \operatorname{Hom}_F(V, F)$ be its dual space, and let $B\colon V \times V \to F$ be a nondegenerate bilinear form. Let $W \subseteq V$ be a subspace and

$$W^\perp := \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}.$$

Show that $V/W^\perp \simeq W^*$.

*Solution.* The bilinear form $B$ induces a map $v \mapsto v^* \in V^*$ where $v^*(x) = B(v, x)$ for $x \in V$. Since $B$ is nondegenerate, this map is an isomorphism. If we further restrict to $W$, we obtain a map $V \to W^*$ by $v \mapsto v^*|_W$. The kernel of this map is $\{v \in V : B(v, w) = 0 \text{ for all } w \in W\} = W^\perp$. The restriction map $V^* \to W^*$ is surjective, since $V = W \oplus W^\perp$ so given $\phi \in W^*$ we can extend by zero on $W^\perp$ to get $\phi \in V^*$. Therefore $V/W^\perp \simeq W^*$.

**Problem 6**. Let $G := \mathbb{Z} \times \mathbb{Z}$ and let $H := \langle (2, 3), (3, 2) \rangle \subseteq G$ be the subgroup generated by $(2, 3)$ and $(3, 2)$. Show that $G/H$ is a cyclic group and compute its order.

*Solution.* There are several ways to do this depending on the experience of the student. The more sophisticated way considers a $\mathbb{Z}$-linear map $T\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given with respect to standard bases by the matrix $A = \begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}$. We are interested in the cokernel, $\mathbb{Z}^2/T(\mathbb{Z}^2)$, which is unaffected by elementary row and column operations which reduces $A$ to $\begin{pmatrix} 1 & 0 \\ 0 & 5 \end{pmatrix}$ from which it follows that $G/H \simeq \mathbb{Z}/5\mathbb{Z}$.

Taking a more pedestrian approach, it is clear that $H$ is also given by $H = \langle (1, -1), (0, 5) \rangle$, and since $G = \mathbb{Z}^2$ has generators $(1, -1), (0, 1)$, it is again quick to deduce the quotient. Concretely, $G/H$ is the cyclic group generated by the coset $(0, 1) + H$.

**Problem 7**. Let $H = \langle \sigma, \tau \rangle \subseteq S_4$ be the subgroup of the symmetric group $S_4$ generated by the elements $\sigma := (1\,2)$ and $\tau := (3\,4)$.

(a) Compute the order of $H$.

(b) Show that $H$ is not a normal subgroup of $S_4$.

(c) Compute the normalizer of $H$ in $S_4$.

*Solution.* For (a), the 2-cycles have order 2 and disjoint cycles commute, so $H$ is a group of order 4 isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

For (b), we know that for any permutation $\rho$, and $r$-cycle $(a_1\, a_2\, \cdots a_r)$, that

$$\rho(a_1\, a_2, \cdots a_r)\rho^{-1} = (\rho(a_1)\, \rho(a_2), \cdots \rho(a_r)).$$

Now observe that $(1\,2\,3\,4)\sigma(1\,2\,3\,4)^{-1} = (2\,3) \notin H$, so $H$ is not normal in $S_4$.

We know that $H \leq N := N_{S_4}(H) \leq S_4$, and since $H$ is not normal, $N_{S_4}(H) \neq S_4$. By Lagrange's theorem, we have $4 = \#H \mid \#N \mid 24$, so $\#N = 4, 8, 12$.

If we let $S_4$ act on its subgroups by conjugation, then the orbit-stabilizer theorem says that $[S_4 : N_{S_4}(H)]$ is equal to the number of conjugates of $H$. Since each group conjugate to $H$ in $S_4$ is generated by a pair of disjoint transpositions, counting we find that there are three conjugates of $H$, hence $\#N = 8$, and it is easy to check that $\rho = (1\,3\,2\,4)$ is in the normalizer, so $N = N_{S_4}(H) = \langle \sigma, \tau, \rho \rangle$ is a dihedral group of order 8.

**Problem 8**. Let $p$ be prime and let $R$ be a ring (with 1) with $\#R = p^2$. Show $R$ is commutative.

*Solution.* The ring $R$ is in particular an abelian group, so either $R \simeq \mathbb{Z}/p^2\mathbb{Z}$ or $R \simeq \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ as abelian groups. In the first case, the multiplication law is unique, since $1 \cdot 1 = 1$ so the rest is determined by distributivity. In the second case, it is determined on the subring $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ generated by 1; let $x \in R$ be not in the span of 1: then $R$ is commutative, because $x$ commutes with itself (and 1), so it commutes with every $ax + b \in R$ (with $a, b \in \mathbb{Z}/p\mathbb{Z}$).

**Problem 9**. Indicate whether each of the statements below is true or false. If true, briefly justify the statement; if false, provide an explicit counterexample.

(a) Let $I \subsetneq \mathbb{Z}[x]$ be a proper ideal satisfying $\langle x\rangle \subseteq I \subsetneq \mathbb{Z}[x]$. Then $I$ is a prime ideal.

(b) In a PID, nonzero prime ideals are maximal.

(c) Let $f(x) \in \mathbb{Q}[x]$ be irreducible. Then $f$ is irreducible in $\mathbb{Q}[x, y]$.

(d) If $p \in \mathbb{Z}$ is a prime, $\langle x^3 - p\rangle$ is a maximal ideal in $\mathbb{Z}[x]$.

(e) $26x^3 + x + 64$ is irreducible in $\mathbb{Z}[x]$.

*Solution.* Statement (a) is false: $\langle x, 4\rangle$ is not prime.

Statement (b) is true: nonzero principal prime ideals in an integral domain are generated by prime, hence irreducible elements, which in a PID generate maximal ideals.

Statement (c): true! Try to factor as polynomials in $\mathbb{Q}[x][y]$ and force the degree in $y$ to be zero, which brings the problem down to $\mathbb{Q}[x]$, and we know $\mathbb{Q}[x, y]^\times = \mathbb{Q}[x]^\times = \mathbb{Q}^\times$.

For (d), we claim it is false. By Eisenstein's criterion, we know that $x^3 - p$ is irreducible in $\mathbb{Q}[x]$, and since primitive, is irreducible in $\mathbb{Z}[x]$. In $\mathbb{Q}[x]$ this would generate a maximal ideal, but not in $\mathbb{Z}[x]$, for example $\langle x^3 - p, n\rangle$ is strictly larger (and still proper) for any $n \geq 2$.

Finally (e) is true: it's a cubic, so we might think about roots, but it is ugly enough not to be tempted. And Eisenstein is out, so we think of reduction criteria. Mod 2 is of no value but in $(\mathbb{Z}/3\mathbb{Z})[x]$ the polynomial is $2x^3 + x + 1$ which has no roots in $\mathbb{Z}/3\mathbb{Z}$ (a field) so is irreducible there, so must be irreducible in $\mathbb{Z}[x]$ (primitive).

**Problem 10**. Let $R \subseteq \mathbb{C}$ be a subring which is finite-dimensional as a $\mathbb{Q}$-vector space. Show that $R$ is a field.

*Solution.* First, $R$ is a domain, since it is a subring of $\mathbb{C}$. We need to show all nonzero elements of $R$ have inverses. Let $a \in R$ be nonzero. Then the map $R \to R$ by $x \mapsto ax$ is $\mathbb{Q}$-linear and injective because $R$ is a domain. Therefore as a map of finite-dimensional $\mathbb{Q}$-vector spaces this map is also surjective, so there exists $b \in R$ such that $ab = 1$. Thus $R$ is a field.