Sample questions for algebra preliminary exam

Problem 1. Let A be a real 2×2 matrix such that $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector with eigenvalue 3 and such that $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector with eigenvalue -2. Compute A^{-1} applied to $\begin{pmatrix} 7 \\ 8 \end{pmatrix}$.

Problem 2. Determine whether the surface in $\mathbb{R}^3$ defined by

$$x^2 + y^2 + z^2 - 2xy - 2xz - 2yz = 1$$

is a ellipsoid (an ellipse rotated about one of its axes) or a hyperboloid (a hyperbola rotated about one of its axes).

Problem 3. Let $n \in \mathbb{Z}_{\geq 1}$, and let V be the vector space of real polynomials of degree at most n. Consider the linear operator $T \colon V \to V$ defined by $T(f)(x) = f(1-x)$.

(a) Compute the determinant of T.

(b) Consider the bilinear form on V defined by $\langle f, g \rangle = \displaystyle\int_0^1 f(x)g(x)\,\mathrm{d}x$. Show that T is self-adjoint with respect to this inner product.

(c) For $n = 2$, find a basis of V consisting of eigenvectors for T.

Problem 4. Let V be a finite-dimensional vector space and let $T \colon V \to V$ be a linear operator satisfying $T^2 = T$.

(a) Show that the only possible eigenvalues of T are zero or one.

(b) If E_λ is the λ-eigenspace, show that $E_0 = \ker T$ and E_1 is the image of T.

(c) Show that T is diagonalizable.

Problem 5. Let V be a finite-dimensional vector space over a field F, let $V^* := \mathrm{Hom}_F(V, F)$ be its dual space, and let $B \colon V \times V \to F$ be a nondegenerate bilinear form. Let $W \subseteq V$ be a subspace and

$$W^\perp := \{v \in V : B(v, w) = 0 \text{ for all } w \in W\}.$$

Show that $V/W^\perp \simeq W^*$.

Problem 6. Let $G := \mathbb{Z} \times \mathbb{Z}$ and let $H := \langle (2,3), (3,2) \rangle \subseteq G$ be the subgroup generated by $(2,3)$ and $(3,2)$. Show that G/H is a cyclic group and compute its order.

Problem 7. Let $H = \langle \sigma, \tau \rangle \subseteq S_4$ be the subgroup of the symmetric group S_4 generated by the elements $\sigma := (1\,2)$ and $\tau := (3\,4)$.

(a) Compute the order of H.

(b) Show that H is not a normal subgroup of S_4.

(c) Compute the normalizer of H in S_4.

Problem 8. Let p be prime and let R be a ring (with 1) with $\#R = p^2$. Show R is commutative.

Problem 9. Indicate whether each of the statements below is true or false. If true, briefly justify the statement; if false, provide an explicit counterexample.

(a) Let $I \subsetneq \mathbb{Z}[x]$ be a proper ideal satisfying $\langle x \rangle \subseteq I \subsetneq \mathbb{Z}[x]$. Then I is a prime ideal.

(b) In a PID, nonzero prime ideals are maximal.

(c) Let $f(x) \in \mathbb{Q}[x]$ be irreducible. Then f is irreducible in $\mathbb{Q}[x, y]$.

(d) If $p \in \mathbb{Z}$ is a prime, $\langle x^3 - p \rangle$ is a maximal ideal in $\mathbb{Z}[x]$.

(e) $26x^3 + x + 64$ is irreducible in $\mathbb{Z}[x]$.

Problem 10. Let $R \subseteq \mathbb{C}$ be a subring which is finite-dimensional as a $\mathbb{Q}$-vector space. Show that R is a field.