Sample solutions to questions for analysis preliminary exam

Problem 1. Let $A \subset \mathbb{R}$ be an open set and let $f\colon A \to \mathbb{R}$ be a function. Give three criteria (ϵ-δ, open sets, sequences) for f to be continuous on A. Show that two of these definitions are equivalent.

Solution. We claim that the following are equivalent:

1. For all $a \in A$ and for all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - a| < \delta$ and $x \in A$ implies $|f(x) - f(a)| < \epsilon$;
2. For all open sets $V \subseteq \mathbb{R}$, the inverse image $f^{-1}(V) \subseteq A$ is open; and
3. For all $a_n \to a \in A$, we have $f(a_n) \to f(a) \in \mathbb{R}$.

First, (1) $\Rightarrow$ (2). Let $V \subseteq \mathbb{R}$ be open, and let $a \in U := f^{-1}(V)$. Since V is open there is an open interval $B_\epsilon(f(a)) = (f(a) - \epsilon, f(a) + \epsilon) \subseteq V$ of $f(a)$, so by (1) we have $f(B_\delta(a)) \subseteq B_\epsilon(f(a)) \subseteq V$; thus $B_\delta(a) \subseteq f^{-1}(B_\epsilon(f(a))) \subseteq U$ is an open neighborhood of a contained in U, so U is open.

Second, (2) $\Rightarrow$ (3). Let $\epsilon > 0$. By (2), we have $U := f^{-1}(B_\epsilon(a))$ open, so there exists an open neighborhood $B_\delta(a) \subseteq U$. Since $a_n \to a$, there exists $N \in \mathbb{Z}_{\geq 0}$ such that $a_n \in B_\delta(a)$ for $n \geq N$. Putting these together, we have $f(a_n) \in B_\epsilon(a)$ for $n \geq N$, which is (3).

Finally, (3) $\Rightarrow$ (1), which we prove by the contrapositive. By the negation of (1), we find that exists $a \in A$ and $\epsilon > 0$ such that for all $\delta = 1/n > 0$ (with $n \in \mathbb{Z}_{>0}$), there exists $a_n \in A$ such that $a_n \in B_\epsilon(\delta)$ but $|f(a_n) - f(a)| \geq \epsilon$. Thus the sequence $a_n \to a$, but $f(a_n) \not\to f(a)$, as desired.

Problem 2. Prove that for all $x > 0$ we have the inequality

$$\sin x > x - \frac{x^3}{6}.$$

Solution. By Taylor's theorem with Lagrange's form of the remainder, letting $f(x) = \sin x$ we have

$$\sin x = x + \frac{f^{(3)}(c)}{3!} x^3$$

for some $0 < c < x$, where $f^{(3)}(x) = (\sin x)''' = -\cos x$ so $f^{(3)}(c) < 1$. The inequality follows.

To do it "by hand", let $f(x) := x - x^3/6$. Then $f'(x) = 1 - x^2/2$ and so f is decreasing for $x > \sqrt{2}$, hence for $x \geq 3$ we have $f(x) < f(3) = -3/2 < -1 < \sin x$. For $0 < x < 3$, consider the Taylor series

$$\sin x - x = \sum_{n=1}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!};$$

it has terms of alternating sign, and since

$$\frac{x^{2n+3}}{(2n+3)!} = \frac{x^2}{(2n+3)(2n+2)} \frac{x^{2n+1}}{(2n+1)!} < \frac{x^{2n+1}}{(2n+1)!}$$

for $n \geq 1$, so we may apply the zig-zag criterion in the alternating series test: we have

$$\sin x - x = -\frac{x^3}{6} + \frac{x^5}{120} - \cdots < -\frac{x^3}{6}$$

since the next term is positive.

Problem 3. Show that if the uniformly continuous functions $f_n\colon \mathbb{R} \to \mathbb{R}$ for $n \geq 1$ converge uniformly to $f\colon \mathbb{R} \to \mathbb{R}$, then f is uniformly continuous.

Solution. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, there exists $N \in \mathbb{Z}_{\geq 1}$ such that for all $x \in \mathbb{R}$ we have $|f_N(x) - f(x)| < \epsilon/3$. Moreover, since the functions f_N are uniformly continuous, there exists $\delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have $|f_N(x) - f_N(y)| < \epsilon/3$. Therefore

$$|f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

the first by uniform convergence at x, the second by uniform continuity of f_N, and the third by uniform convergence at y. Thus f is uniformly continuous.

Problem 4. Let (X, d) be a compact metric space and $f\colon X \to X$ be a continuous function such that if $x \neq y$, then $d(f(x), f(y)) < d(x, y)$. Show that f has a unique fixed point.

Solution. Consider the function

$$\begin{aligned} g\colon X &\to \mathbb{R}_{\geq 0} \\ x &\mapsto g(x) = d(x, f(x)). \end{aligned}$$

The map g is continuous, since d and f are continuous; since X is compact, by the extreme value theorem g attains its minimum at some point x. Let $y := f(x)$. If $x \neq y$, then

$$g(y) = d(y, f(y)) = d(f(x), f(f(x))) < d(x, f(x)) = g(x);$$

this contradicts that the minimum of g is obtained at x. Thus $x = y = f(x)$, so x is a fixed point. To show uniqueness, suppose $x' \in X$ has $f(x') = x'$. If $x' \neq x$, then $d(x, x') = d(f(x), f(x')) < d(x, x')$, a contradiction. So $x' = x$, and the fixed point is unique.

Problem 5. Let U be a connected, open subset of $\mathbb{R}^n$. Suppose $f\colon U \to \mathbb{R}$ is a function that is differentiable on U and that all partial derivatives $\dfrac{\partial f}{\partial x_i}(p) = 0$ vanish for all $p \in U$. Prove that f is constant.

Solution. We first prove this in the special case where U is open convex. Let $p, q \in U$ and define $g\colon [0, 1] \to \mathbb{R}$ by $g(t) := f(x(t))$, with $x(t) = (x_i(t))_i := (1 - t)p + tq \in U$ for $t \in [0, 1]$ since U is convex. By the chain rule, for all $t \in (0, 1)$ we have

$$g'(t) = \frac{\mathrm{d}g}{\mathrm{d}t}(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x(t)) \frac{\mathrm{d}x_i}{\mathrm{d}t}(t) = 0$$

because all partial derivatives vanish at all points in U. By the mean value theorem, there exists $c \in (0, 1)$ such that

$$g(1) - g(0) = g'(c);$$

but $g(1) = f(q)$ and $g(0) = f(p)$, so

$$f(q) - f(p) = g'(c) = 0$$

and hence $f(q) = f(p)$.

Finally, choose $p_0 \in U$, and let $W := \{p \in U : f(p) = f(p_0)\}$. Then W is closed (it is the inverse image of $f(p)$) and nonempty. It is also open: if $p \in W$, then in any open (convex) ball V of p in U, by the previous paragraph we have $f(q) = f(p) = f(p_0)$ for all $q \in V$, hence $V \subseteq W$. Since U is connected, we conclude that $W = U$ and f is constant.

Problem 6. Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}_{>0}$ be a monotone, decreasing function defined on the positive real numbers with

$$\int_0^\infty f(x)\,\mathrm{d}x < \infty.$$

Show that

$$\lim_{x\to\infty} xf(x) = 0.$$

Solution. Since f is monotone decreasing, we obtain a lower bound on the integral using a Riemann sum with right endpoints:

$$\sum_{n=1}^{\infty} nf(n) < \int_0^\infty f(x)\,\mathrm{d}x < \infty.$$

Of course if a series of positive terms converges, then its terms tend to 0, so $\lim_{n\to\infty} nf(n) = 0$. Let $\epsilon > 0$. Then there exists $X \in \mathbb{R}_{>0}$ such that whenever $x \geq X$, we have $f(x) < \epsilon/2$. Similarly, there exists $N \in \mathbb{Z}_{\geq 0}$ such that whenever $n \geq N$ we have $nf(n) < \epsilon/2$. Thus whenever $x \geq \max(N, X)$, letting $n := \lfloor x \rfloor \leq x$ we have

$$xf(x) \leq xf(n) = (x - n + n)f(n) \leq f(n) + nf(n) < \epsilon/2 + \epsilon/2 < \epsilon.$$

Thus $\lim_{x\to\infty} xf(x) = 0$.

Problem 7. Suppose that X and Y are topological spaces with Y compact, and give $X \times Y$ the product topology. Show that the projection map $\pi\colon X \times Y \to X$ is a closed map.

Solution. Let $Z \subseteq X \times Y$ be closed; we show that $X \smallsetminus \pi(Z)$ is open. Let $x \in X$ have $x \notin \pi(Z)$. Then $\{x\} \times Y$ is contained in $X \times Y \smallsetminus Z$. By the tube lemma, one can find an open set $V \subseteq X$ containing x such that $V \times Y \subseteq X \times Y \smallsetminus Z$. Thus $V \subseteq X$ is in the complement of $\pi(Z)$, showing $X \smallsetminus \pi(Z)$ is open.

Here is a direct proof. Again, let $Z \subseteq X \times Y$ be closed, and let $x \notin \pi(Z)$. Then $(x, y) \in (X \times Y) \smallsetminus Z$ for all $y \in Y$. Since $(X \times Y) \smallsetminus Z$ is open, for each $y \in Y$ there exists an open subset $U_y \times V_y \subseteq (X \times Y) \smallsetminus Z$ containing (x, y). The collection of open sets $\{V_y\}_{y\in Y} \subseteq Y$ form an open cover. Since Y is compact, this reduces to an open cover with $Y = \bigcup_{i=1}^{r} V_{y_i}$. Let $U := \bigcap_{i=1}^{k} U_{y_i}$. Then $x \in U$. And if $x' \in U$, then

$$\{x'\} \times \{V_{y_i}\} \subseteq U_{y_i} \times V_{y_i} \subseteq (X \times Y) \smallsetminus Z$$

for all i. Thus $\{x'\} \times Y \subseteq (X \times Y) \smallsetminus Z$, and so $U \subseteq X \smallsetminus \pi(Z)$ is open, as claimed.

Problem 8. Give an example of a Hausdoff topological space X and an equivalence relation $\sim$ on X so that the topological space $Y = X/\sim$ is not Hausdorff.

Solution. We use the line with a doubled origin. Let $X := \{(x, i) \in \mathbb{R} : i \in \{1, 2\}\}$. Define an equivalence relation on X by $(x, i) \sim (x', i')$ when $x = x' \neq 0$ and $i \neq i'$. It is straightforward to check that this is an equivalence relation, and the quotient $Y := X/\sim$ has equivalence classes $[(0, 1)] = \{(0, 1)\}$, $[(0, 2)] = \{(0, 2)\}$, and $[(x, 1)] = [(x, 2)] = \{(x, 1), (x, 2)\}$ for $x \neq 0$. The neighborhoods of $(0, i)$ are open intervals in $\mathbb{R} \times \{i\}$ containing 0, so any two neighborhoods of $[(0, 1)]$ and $[(0, 2)]$ intersect.

Problem 9. Prove or disprove: the set $\mathbb{Q}$ of rational numbers is the intersection of a countable family of open subsets of $\mathbb{R}$.

Solution. The statement is false. We have

$$\mathbb{R} \setminus \mathbb{Q} = \bigcap_{a \in \mathbb{Q}} (\mathbb{R} \setminus \{a\}).$$

Suppose that $\mathbb{Q} = \bigcap_n G_n$ with each $G_n \subseteq \mathbb{R}$ open. Since $\mathbb{Q}$ is dense in $\mathbb{R}$, and $\mathbb{Q} \subseteq G_n$ we have G_n open dense in $\mathbb{R}$ for all n. Thus

$$\emptyset = \mathbb{Q} \cap (\mathbb{R} \setminus \mathbb{Q})$$

is a countable intersection of open dense sets. This contradicts the Baire category theorem, which says that any countable intersection of open dense sets is dense.