Combinatorial Results from Dynamical Systems

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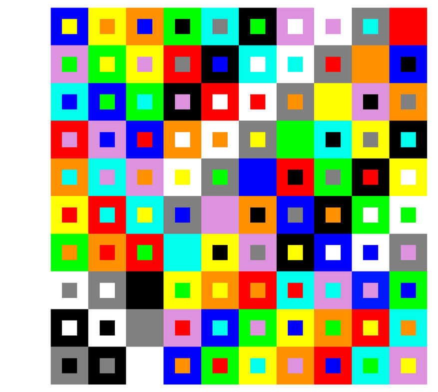


The Idea:

We start with a dynamical system and study the patterns realized by their orbits. In our case, we consider a family of maps called the signed shifts. The patterns realized by their periodic orbits can be shown to be in bijection with the cyclic permutations in a pattern-avoiding class.

Signed Shifts

Definition

Fix $k \geq 2$. Let $\mathcal{W}_k$ be the set of words $s = s_1 s_2 \cdots$ over the alphabet $\{0, 1, \cdots, k-1\}$ with the lexicographic ordering. Fix $\sigma = \sigma_0\sigma_1 \cdots \sigma_{k-1} \in \{+,-\}^k$. In addition, let $T_\sigma^+ = \{t : \sigma_t = +\}$ and $T_\sigma^- = \{t : \sigma_t = -\}$, and notice that these sets form a partition of the alphabet $\{0, 1, \ldots, k-1\}$. Then we can define the **signed shift** $\Sigma_\sigma : \mathcal{W}_k \to \mathcal{W}_k$ to be

$$\Sigma_\sigma(s_1 s_2 s_3 \cdots) = \begin{cases} s_2 s_3 \cdots & \text{if } s_1 \in T_\sigma^+ \\ \bar{s}_2 \bar{s}_2 \cdots & \text{if } s_1 \in T_\sigma^- \end{cases}$$

where $\bar{s}_i = k - s_i + 1$. If $\sigma = + + \cdots +$, then Σ_σ is the traditional left shift on k letters.

Alternate Definition

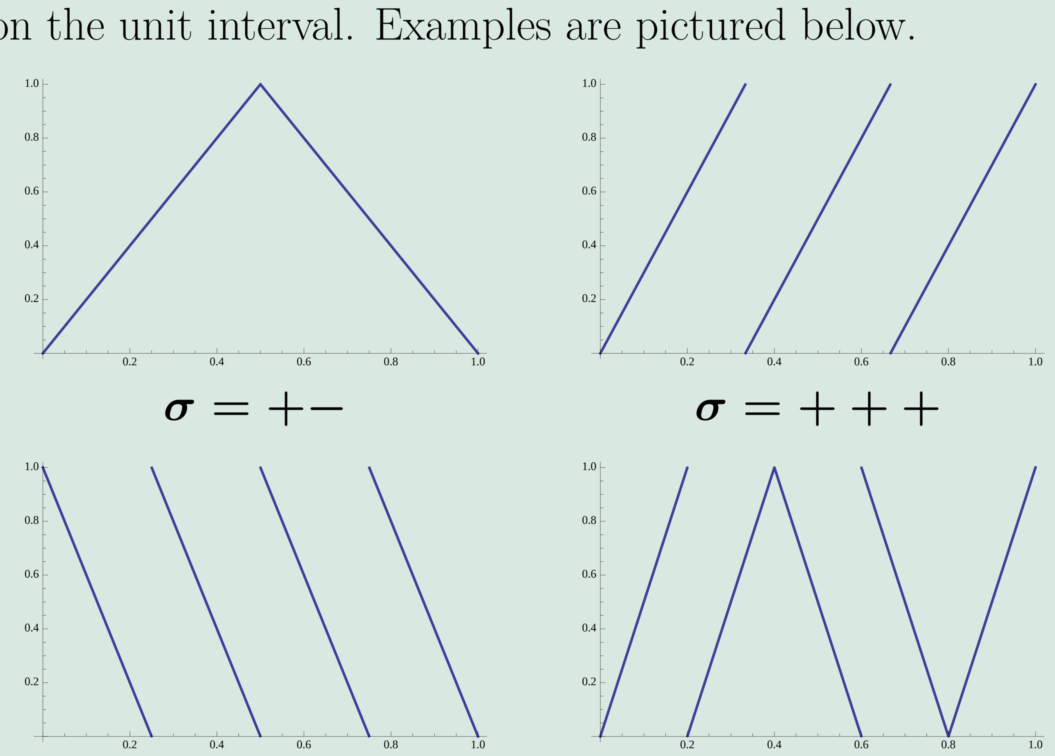
For our purposes, it is more convenient to use the following equivalent definition of Σ_σ. We can define Σ_σ to be the left shift on the set of words

$$\Sigma_\sigma(s_1 s_2 s_3 \cdots) = s_2 s_3 \cdots$$

where the set of words is endowed with a special ordering $\prec_\sigma$, dependent on σ. When $\sigma = ++\cdots+$, this new ordering is the same as the lexicographic ordering.

Properties

Signed shifts are widely studied in dynamical systems. Each is order-isomorphic to a **signed sawtooth map** defined on the unit interval. Examples are pictured below.



$\sigma = +-$ $\quad$ $\sigma = +++$ $\quad$ $\sigma = - - - - -$ $\quad$ $\sigma = + + - - +$

For every σ, the signed shift Σ_σ (and the corresponding sawtooth map) is a chaotic map and so, for every $n \geq 1$, there is a periodic orbit of length n associated to Σ_σ.

Using the alternate definition, it is clear that the periodic points of Σ_σ are words of the form $s = (s_1 \cdots s_n)^\infty$ where $s_1 \cdots s_n$ is primitive.

Patterns of Dynamical Systems

Definition

Given a linearly ordered set X and a map $f : X \to X$, the orbit of $x \in X$ is the sequence

$$x, f(x), f^2(x), f^3(x), \cdots.$$

If there are no repetitions among the first n elements of the orbit, we define the **pattern** of length n of f at x, denoted $\mathrm{Pat}(x, f, n)$, to be a permutation π where π is in the same relative order as the first n elements of the orbit. We call the set of all patterns realized by a map f the set of **allowed patterns**.

Previously Known Results

In [2], the allowed patterns of the k-shift (Σ_σ where $\sigma = +^k$) were characterized and enumerated. In [3], another family of maps called β-shifts were studied. In addition, determining the asymptotic growth of the number of allowed patterns of a map f allows one to compute the **topological entropy** of f, a measure of chaos from dynamical systems. More information can be found in [1].

Example

If f is the logistic map, defined

$$f(x) = 4x(1-x)$$

on the unit interval, then the orbit of the point $x = 0.8$ is

$$0.8, 0.64, 0.922, 0.289, 0.822, \cdots.$$

Therefore the pattern of length 5 is $\mathrm{Pat}(.8, f, 5) = 32514$.

Periodic Patterns

Given a periodic point $x \in X$ with period n, we can define the **periodic pattern** to be

$$\mathrm{PP}(x, f) = \mathrm{Pat}(x, f, n).$$

We'll denote the set of periodic patterns as $\mathcal{P}(f)$ and the set of periodic patterns of length n as $\mathcal{P}_n(f)$. For a periodic pattern π, we denote its equivalence class

$$[\pi] = \{\pi_i \pi_{i+1} \cdots \pi_n \pi_1 \cdots \pi_{i-1} : 1 \leq i \leq n\}.$$

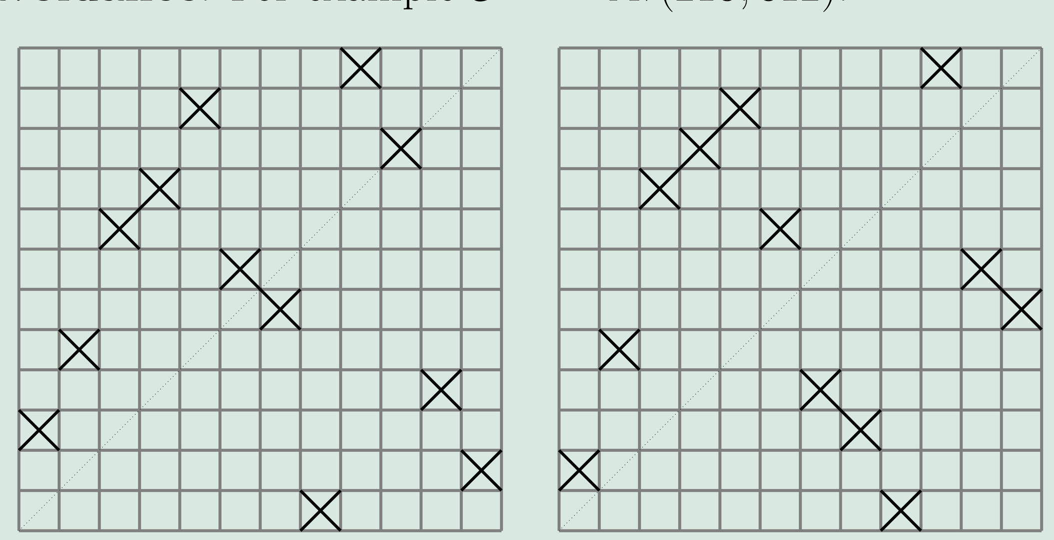
If $\pi = \mathrm{PP}(x, f)$, then $[\pi]$ is the set of patterns realized by all points in the orbit of x. We denote the set of equivalence classes by $\overline{\mathcal{P}}(f)$ and define $p_n(f) = |\overline{\mathcal{P}}(f)| = |\mathcal{P}(f)|/n$.

The σ-Class and the Bijection θ

The σ-class

For $\sigma \in \{+,-\}^k$, we say a permutation π is in the **σ-class, $\mathcal{S}^\sigma$,** if it can be written as the concatenation $\pi = \alpha_0\alpha_1 \cdots \alpha_{k-1}$ where α_i is increasing if $\sigma_i = +$ and decreasing if $\sigma_i = -$. For example, $\mathcal{S}^{+-}$ is the set of unimodal permutations, those that are increasing then decreasing. Below are examples of permutations from $\mathcal{S}^{+--}$. The left permutation is obtained by concatenating the increasing sequence 3 5 8 9 11 and the decreasing sequences 7 6 1 and 12 10 4 2.

These σ-classes can be characterized in terms of **pattern avoidance**. For example $\mathcal{S}^{+-} = \mathrm{Av}(213, 312)$.



The map $\theta : \mathcal{S}_n \to \mathcal{C}_n$

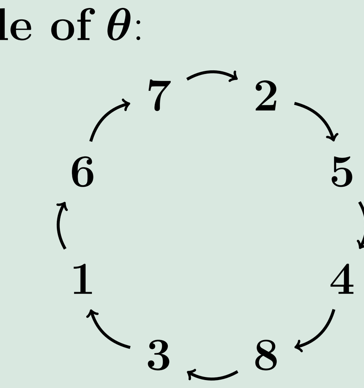
We define the map θ from the set of permutations to the set of cyclic permutations as $\theta(\pi) = \hat{\pi}$ where $\pi = \pi_1\pi_2 \cdots \pi_n$ in one line notation and $\hat{\pi} = (\pi_1 \cdots \pi_n)$ in cycle notation. Notice that if π and π' are in the same equivalence class, then $\hat{\pi} = \hat{\pi}'$.

Example of θ:

$\pi = 72548316$

θ

$\hat{\pi} = 65184723$



Bijection

The θ provides a bijection $\overline{\mathcal{P}}(\Sigma_\sigma) \to \mathcal{C}_n \cap \mathcal{S}^\sigma$ when $\sigma \neq -^k$. For example, the periodic patterns of Σ_{+-} (the tent map) are in bijection with unimodal cyclic permutations.

As an example of the bijection, consider $\sigma = +++$ (so we can use lexicographic ordering). If $s = (0112210)^\infty$, then $\pi = \mathrm{PP}(s, \Sigma_{+++}) = 2457631$. Therefore, applying θ, we get

$$\hat{\pi} = \theta(\pi) = (2457631) = 2415736.$$

Note $\hat{\pi}$ can be broken up into three increasing segments: 24, 157, 36, and so is in $\mathcal{S}^{+++}$. The length of these segments is determined by the primitive part of s, which has two 0s, three 1s, and two 2s. This will generally occur.

In the reverse direction, if we are given a cyclic permutation $\hat{\pi}$ in a given σ-class, we break it up into appropriate increasing and decreasing segments (made possible by its inclusion in $\mathcal{S}^\sigma$), and the lengths of these give the number of 0s, 1s, etc. in $s_1 \cdots s_n$. After taking a representative $\pi \in [\pi] = \theta^{-1}(\hat{\pi})$, we can reconstruct a periodic word whose pattern is π. Therefore, π must be a periodic pattern.

Enumeration of Periodic Patterns

For special cases, we enumerate $\overline{\mathcal{P}}(\Sigma_\sigma)$ and via θ, we see there are interesting combinatorial results.

The tent map

The signed shift Σ_{+-} is often called the **tent map**.

$$p_n(\Sigma_{+-}) = \frac{1}{2n} \sum_{\substack{d|n \\ d \text{ odd}}} \mu(d) 2^{\frac{n}{d}}$$

where μ denotes the Möbius function. Because these patterns are in bijection with the unimodal cyclic permutations, this provides an enumeration for them as well.

The k-shift

When $\sigma = +^k$, we call Σ_σ the **k-shift** and denote it Σ_k. Let $C(n,k) = p_n(\Sigma_k) - p_n(\Sigma_{k-1})$. By bijection θ, $C(n,k)$ is also the number of cyclic permutations of length n with exactly $k-1$ descents. We have the following recurrence.

$$C(n,2) = \frac{1}{n} \sum_{d|n} \mu(d) 2^{\frac{n}{d}}$$

and

$$C(n,k) = \frac{1}{n} \sum_{d|n} \mu(d) k^{\frac{n}{d}} - \sum_{i=2}^{k-1} \binom{n+k-i}{k-i} C(n,i).$$

This agrees with the generating function for cyclic permutations of length n with $k-1$ descents given in [4]. The number of periodic patterns for Σ_k is

$$p_n(\Sigma_k) = \sum_{i=2}^{k} C(n,i).$$

We can also enumerate the number of periodic patterns when $\sigma = -^k$. We call this map the **reverse k-shift** and denote it by Σ_k^-. When $n \neq 2 \bmod 4$, $p_n(\Sigma_k^-) = p_n(\Sigma_k)$ and there is a simple bijection $\mathcal{P}(\Sigma_k^-) \to \mathcal{P}(\Sigma_k)$ involving words. Even though θ fails as a bijection when $n = 2 \bmod 4$, we can still enumerate cyclic permutations with exactly $k-1$ ascents and the periodic patterns for this case.

Pattern Avoidance

This research addresses the question of enumerating the cyclic permutations in certain pattern avoiding classes. For most pattern avoiding classes, even seemingly simple ones, this question is still open.

References

[1] J.M. Amigó, *Permutation complexity in dynamical systems*, Springer Series in Synergetics, Springer- Verlag, Berlin, 2010.

[2] S. Elizalde, The number of permutations realized by a shift, *SIAM J. Discrete Math.* 23 (2009), 765Ð786.

[3] S. Elizalde, Permutations and β-shifts, *J. Combin. Theory Ser.* A 118 (2011), 2474Ð2497.

[4] I. Gessel and C. Reutenauer, Counting Permutations with Given Cycle Structure and Descent Set. *Journal of Combinatorial Theory.* Series A 64, (1993) 189-215.