Avoiding Geometric Progressions in the Integers

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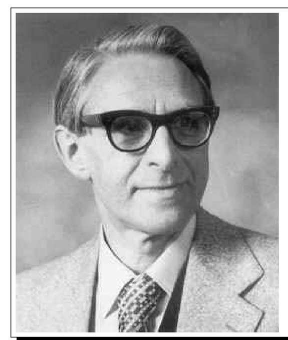
Definitions

A 3-term **arithmetic progression** (**AP**) is a set $\{a, a+b, a+2b\}$. (i.e. $\{2,4,6\}$ or $\{5,8,11\}$)
A 3-term **geometric progression** (**GP**) is a set $\{a, ar, ar^2\}, r \in \mathbb{Q}$. (i.e. $\{3,9,27\}$, $\{5,10,20\}$ or $\{4,6,9\}$)
The **density** of a set $A \subset \mathbb{N}$, denoted $d(A)$ can be thought of as the percentage of the integers contained in A. Since this is not always well defined, we also define the **upper density** $\bar{d}(A)$. More rigorously,

$$d(A) = \lim_{N\to\infty} \frac{|A \cap [1,N]|}{N} \qquad \bar{d}(A) = \limsup_{N\to\infty} \frac{|A \cap [1,N]|}{N}$$

1. Avoiding Arithmetic Progressions in the Integers

Theorem 1 (Van der Waerden, 1927). *Any coloring of the integers using a finite number of colors will contain monochromatic arithmetic progressions of every length.*



Klaus Friedrich Roth (1925-) is a German-born British mathematician best known for his work in the field of Diophantine approximation, or how well irrational numbers can be approximated by fractions. He was awarded the Fields Medal, the most prestigious award in mathematics, for this work in 1958.

Theorem 2 (Roth, 1953). *Any subset $A \subset \mathbb{N}$ that has positive upper density, $\bar{d}(A) > 0$, contains infinitely many 3-term arithmetic progressions.*
Later generalized by Szemerédi (1975) to progressions of arbitrary length.

2. The greedy AP-free set and lower bounds

What is the largest subset of $[1,N]$ that avoids Arithmetic Progressions?

First try: Greedy set, A_3^*. Include n in A_3^* if n does not create a 3-term-AP in A_3^*.

$$\mathbf{A_3^*} = \{0,1,3,4,9,10,12,13,27,28,30,31\ldots\}$$
$$= \{n \geq 0 \mid n \text{ has no digit 2 in its base 3 representation}\}$$
$$|A_3^* \cap [1,N]| \approx N^{\log_2 3}$$

One can do much better. It is possible to find subsets of $[1,N]$ free of 3-term-APs of size:

$$\frac{1}{\log^{1/4} N} \cdot \frac{N}{2^{2\sqrt{2\log_2 N}}} \text{ (Behrend, 1946)}$$
$$\frac{N \log^{1/4} N}{2^{2\sqrt{2\log_2 N}}} \text{ (Elkin, 2008)}$$

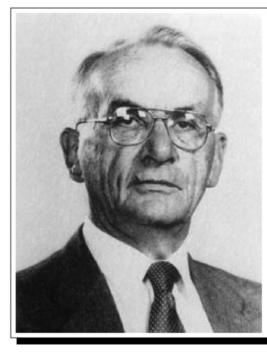
3. Upper bounds of sets free of arithmetic progressions

For sufficiently large N, there exists a 3-term AP in any subset of $[1,N]$ of size:

- $\frac{N}{\log\log N}$ (Roth, 1954)
- $\frac{N}{\log^c N}$ for some constant $c > 0$ (Heath-Brown, 1987)
- $\frac{N}{\log^{1/20} N}$ (Szemerédi, 1990)
- $\frac{N(\log\log N)^{1/2}}{\log^{1/2} N}$ (Bourgain, 1999)
- $\frac{N(\log\log n)^2}{\log^{2/3} N}$ (Bourgain, 2008)
- $\frac{N(\log\log N)^5}{\log N}$ (Sanders, 2010)

4. Rankin's geometric progression free set

In 1961, Rankin suggested looking at sets free of geometric progressions. Because the set of square free numbers, S, is free of geometric progressions, and $d(S) = \frac{6}{\pi^2} \approx 0.6079$ Roth's theorem is false for geometric progressions.



Robert Alexander Rankin (1915-2001) was a Scottish mathematician known for his work in modular forms and the distribution of prime numbers. During World War II his career was interrupted to work on rockets for the British army. In 1939 he developed the Rankin-Selberg method of analytically continuing certain L-functions.

If $\{a,b,c\}$ is a geometric progression, then for every prime, p, $\{v_p(a), v_p(b), v_p(c)\}$ forms an arithmetic progression. Using this, Rankin constructs the 3-term GP-free set

$$\mathbf{G_3^*} = \{n \in \mathbb{N} : \text{for all primes } p, v_p(n) \in A_3^*\}$$
$$= \{1,2,3,5,6,7,8,10,11,13,14,15,16,17,19\ldots\}.$$

Rankin's set is also the greedy set obtained by greedily including integers without creating a geometric progression. Its density is

$$d(G_3^*) = \prod_p \left(\frac{p-1}{p} \sum_{i \in A_3^*} \frac{1}{p^i}\right) = \frac{1}{\zeta(2)} \prod_{i>0} \frac{\zeta(3^i)}{\zeta(2\cdot 3^i)} > 0.71974.$$

What is the greatest possible density of a geometric progression free set?

5. Bounds on the density of sets avoiding geometric progressions

Define:
$$\overline{\alpha} = \sup\{\bar{d}(A) : A \subset \mathbb{N} \text{ is GP-free}\}$$
$$\alpha = \sup\{d(A) : A \subset \mathbb{N} \text{ is GP-free and } d(A) \text{ exists}\}$$

Theorem 3. *We have* $0.71974 < \alpha \leq \overline{\alpha} \leq \frac{7}{8} = 0.875$.

Proof. For any N, let $k \leq N/4$ be odd. A GP-free set cannot contain $k, 2k$ and $4k$. These triples do not overlap, so at least $N/8$ numbers up to N must be excluded. □

The upper bound for the upper density of a GP-free set has been improved several times.

- $\overline{\alpha} \leq \frac{6}{7} \approx 0.8571$ (Riddell, 1969; Beiglböck, Bergelson, Hindman and Strauss, 2006)
- $\overline{\alpha} < 0.8688$ (Brown and Gordon, 1996)
- $\overline{\alpha} < 0.8495$ (Nathanson and O'Bryant, 2013)
- $\overline{\alpha} < 0.8339$ (Claimed by Riddell, 1969 but stated "The details are too lengthy to be included here.")

Theorem 4 (M., 2013). *The constant $\overline{\alpha}$, the greatest possible upper density of a 3-term GP-free set, is effectively computable and satisfies*

$$\mathbf{0.730027 < \overline{\alpha} < 0.772059}.$$

6. Avoiding *s*-smooth progressions

Say that a geometric progression $\{a, ar, ar^2\}$ is s-smooth if the common ratio $r \in \mathbb{Q}$, involves only primes at most s. Then define

$$\overline{\alpha_s} = \sup\{\bar{d}(A) : A \subset \mathbb{N} \text{ is free of } s\text{-smooth rational GPs}\}.$$

Key Idea: the first seven 3-smooth numbers, $\{1,2,3,4,6,8,9\}$, contain 4 GPs: $(1,2,4)$, $(2,4,8)$, $(1,3,9)$ and $(4,6,9)$ which cannot all be avoided by removing any single number.

7. Computations

In general: Compute the largest subset of the 3-smooth integers up to k free of GPs. If an additional number must be excluded to avoid 3-smooth GPs, we get a better upper bound for $\overline{\alpha_3}$.

k	# of exclusions	k	# of exclusions	k	# of exclusions	k	# of exclusions
4	1	128	10	576	19	2048	28
9	2	144	11	729	20	2304	29
16	3	192	12	864	21	2592	30
18	4	243	13	972	22	3072	31
32	5	256	14	1024	23	3888	32
36	6	288	15	1296	24	4096	33
64	7	384	16	1458	25	4374	34
81	8	486	17	1728	26	5184	35
96	9	512	18	1944	27	5832	36

$$\overline{\alpha_3} < 1 - \frac{1}{3}\left(\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{18} + \frac{1}{32} + \cdots + \frac{1}{5832}\right) \approx 0.791266$$

This argument can also be made constructive, giving us the following bounds:

$$0.790470 < \overline{\alpha_3} < 0.791266$$
$$0.766513 < \overline{\alpha_5} < 0.775755$$
$$0.734133 < \overline{\alpha_7} < 0.772059$$

8. Computing $\overline{\alpha}$

We can use lower bounds for $\overline{\alpha_s}$ to create GP-free sets with greater upper density than Rankin's set.

Key Idea: Use the $\overline{\alpha_s}$ construction for primes at most s, and stitch this together with Rankin's construction for primes greater than s.

Theorem 5 (M., 2013).

$$\overline{\alpha_s} \prod_{p>s} \left(\frac{p-1}{p} \sum_{i \in A_3^*} p^{-i}\right) \leq \overline{\alpha} \leq \overline{\alpha_s}$$

So, $\lim_{s\to\infty} \overline{\alpha_s} = \overline{\alpha}$. Using this we can compute $\overline{\alpha}$ to within ϵ, for any $\epsilon > 0$, in time

$$O\left(1.6538^{(-2\log_2 \epsilon)^{\frac{1}{\epsilon}}}\right).$$

Using $s = 7$ we get $0.730027 < \overline{\alpha} < 0.772059$.

Primary references

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