Embedding the Complete Bipartite Graph

Jacob Marchman ‘18. Advisors: Peter Doyle and Emma Hartman
Department of Mathematics, Dartmouth College

Introduction

Ringer (1974) and later Bouchet (1978) have shown that the orientable genus of the complete bipartite graph $K_{m,n}$ is $\lceil(m-2)(n-2)/4\rceil$ for $m \geq 2$ and $n \geq 2$ while the nonorientable genus is $\lceil(m-2)(n-2)/2\rceil$ for $m \geq 3$ and $n \geq 3$. Here we show how $K_{m,n}$ can be drawn on the fundamental domains of their lowest-genus surfaces when $(m-2)(n-2)/4$ or $(m-2)(n-2)/2$ are integers. We will use Conway's notation for surfaces where O^w is the sphere with w handles and X^z is the sphere with z cross-caps.

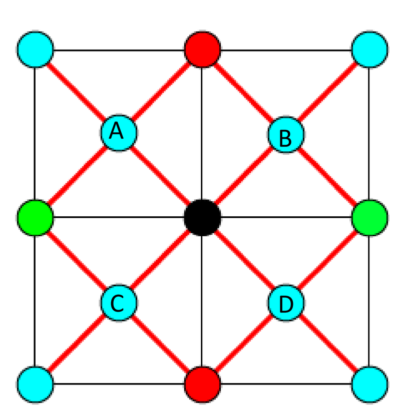
	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	1	1	2	2	2	2	3	3	3	3
4	0	1	1	2	2	3	3	4	4	5	5	6	6
5	0	1	2	3	3	4	5	6	6	7	8	9	9
6	0	1	2	3	4	5	6	7	8	9	10	11	12
7	0	2	3	4	5	7	8	9	10	12	13	14	15
8	0	2	3	5	6	8	9	11	12	14	15	17	18
9	0	2	4	6	7	9	11	13	14	16	18	20	21
10	0	2	4	6	8	10	12	14	16	18	20	22	24

1	2	3	4	5	6	7	8	9	10	11	12	13	14
2	0	0	0	0	0	0	0	0	0	0	0	0	0
3	0	1	1	2	2	3	3	4	4	5	5	6	6
4	0	1	2	3	4	5	6	7	8	9	10	11	12
5	0	2	3	5	6	8	9	11	12	14	15	17	18
6	0	2	4	6	8	10	12	14	16	18	20	22	24
7	0	3	5	8	10	13	15	18	20	23	25	28	30
8	0	3	6	9	12	15	18	21	24	27	30	33	36
9	0	4	7	11	14	18	21	25	28	32	35	39	42
10	0	4	8	12	16	20	24	28	32	36	40	44	48

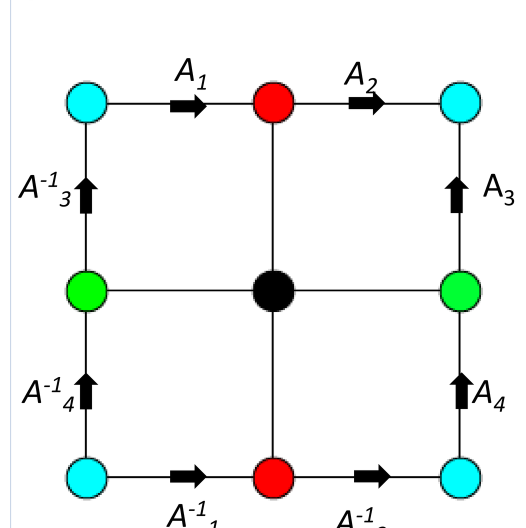
Orientable (left) and nonorientable (right) genus of $K_{m,n}$ for small m,n. Cells are green if $(m-2)(n-2)/4$ or $(m-2)(n-2)/2$ is an integer, blue if the lowest-genus embedding can be recovered by deleting a vertex from the embedding of $K_{m,n+1}$, and red if $(m-2)(n-2)/4$ or $(m-2)(n-2)/2$ is not an integer.

Gluing up the Surface

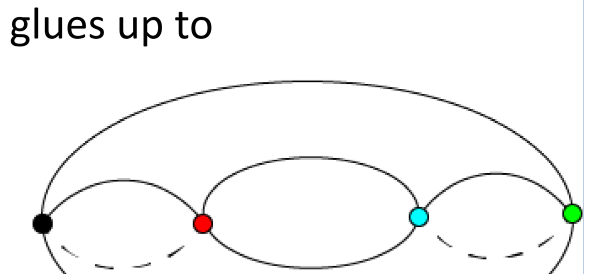
We represent $K_{m,n}$ on the fundamental domain of its surface by drawing m n-sided regular polygons meeting at a point, with each of the m polygons having a vertex at its center and n distinct vertices on its edge. For example, below is $K_{4,4}$. In subsequent pictures, we won't draw the interior vertices.



In order identify the fundamental domain above as O^1, we need to specify a way to glue up the edges. An edge will be labelled A if it is glued in the clockwise direction relative to the border of the domain, and A^{-1} if it is to be glued in counterclockwise direction. Hence,



glues up to



where each polygon becomes a face of the torus.

Orientable Embedding of $K_{4,2n}$ on O^{n-1}

The fundamental domain for $K_{4,2n}$ on its lowest-genus orientable surface O^{n-1} consists of 4 $2n$-gons meeting at a central point. In clockwise orientation, the external edges of the $2n$-gons, $P_1,\ldots,P_4$ are

P_1: $A_1 A_2 A_3 \ldots A_{2n-2}$
P_2: $A_{2n-1}\, A^{-1}_{2n-3}\, A_{2n}\, A^{-1}_{2n-5} A_{2n+1}\, A^{-1}_{2n-7} \ldots A_{3n-3}\, A^{-1}_{1}$
P_3: $A_{3n-2}\, A^{-1}_{3n-3}\, A_{3n-1}\, A^{-1}_{3n-4} A_{3n}\, A^{-1}_{3n-5} \ldots A_{4n-4}\, A^{-1}_{2n-1}$
P_4: $A_{2n-2}^{-1} A^{-1}_{4n-4}\, A^{-1}_{2n-4} A^{-1}_{4n-6}\, A^{-1}_{2n-6} A^{-1}_{4n-8} \ldots A^{-1}_{3n-2}$.

Nonorientable Embedding of $K_{4,2n}$ on X^{2n-2}

The fundamental domain for $K_{4,2n}$ X^{2n-2} consists of 4 $2n$-gons. Its edge gluing turns out to be the same as the orientable case with just three modifications:

1)Swap the 2nd edge of P_1 with the $2n$-3^{rd} edge of P_2 and reverse both the directions of their gluing
2)Swap the $2n$-4^{th} edge of P_2 with the $2n$-4^{th} edge in P_4
3) Swap the 4^{th} edge of $P3$ with the $2n$-6^{th} edge of P_4 and reverse both the directions of their gluing.

Nonorientable Embedding of $K_{4,2n+1}$ on X^{2n-1}

The fundamental domain for $K_{4,2n}$ on X^{2n-2} consists of 4 $2n$-1-gons, $P_1,\ldots,P_4$ whose external edges are:

P_1: $A_1 A_2\, A_3 \ldots A_{2n-1}$
P_2: $A_{2n}\, A^{-1}_{2n-2}\, A_{2n+1} A^{-1}_{2n-4} A_{2n+2} A^{-1}_{2n-6} \ldots A_{3n-1}$ then $A_{3n}\, A^{-1}_{1}$\`
P_3: $A_{3n+1} A_2$ then $A^{-1}_{3n-1} A_{3n+2} A^{-1}_{3n-3} A_{3n+3} A^{-1}_{3n-5} A_{3n+4} \ldots A_{4n-2}\, A^{-1}_{2n}$
P_4: $A_{2n-1}^{-1} A^{-1}_{4n-2}\, A^{-1}_{2n-3} A^{-1}_{4n-2} \ldots A^{-1}_{3n+2} A^{-1}_{3}$ then $A_{3n}\, A^{-1}_{3n+1}$.

A_1

Nonorientable Embedding of $K_{3,4n+2}$ on X^{2n}

The fundamental domain for $K_{3,4n+2}$ on X^{2n} consists of three 4n-gons, P_1, P_2, P_3 with the following edges (listed counterclockwise):

P_1: A_3 then $B_1 B_2 B_3 \ldots B_{4n-2}$ then A_1
P_2: $A_2 A_3$ then $D_1\, B^{-1}_{4n-2} D_2\, B^{-1}_{2} D_3\, B^{-1}_{4n-4} D_4\, B^{-1}_{4}\, D_5\, B^{-1}_{4n-6} \ldots D_{2n-1}\, B^{-1}_{2n}$
P_3: $B_{2n-1}\, D^{-1}_{n-2}\, B^{-1}_{2n+1}\, D^{-1}_{n-1}\, B^{-1}_{2n-3}\, D^{-1}_{n-4}\, B^{-1}_{2n+3}\, D^{-1}_{2n-3}\, B^{-1}_{2n-3}\, D^{-1}_{n-6}$ $\ldots B^{-1}_{1} D_1$ then $A_1 A_2$.

Orientable Embedding of $K_{3,4n+2}$ on O^n

The fundamental domain uses the same gluing as the nonorientable version on X^{2n} except with the following changes:

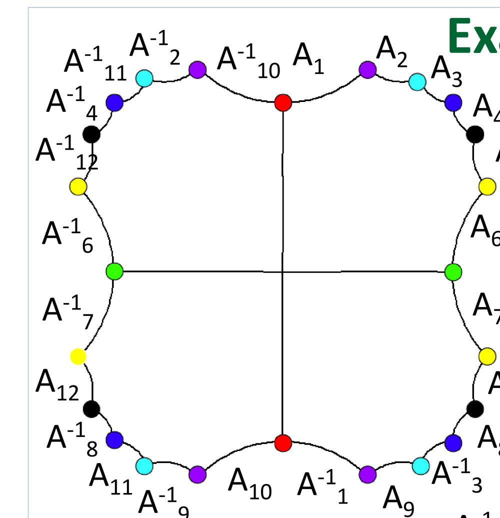
1) In P_1, change the gluing direction of A_1
2) In P_2, change the gluing direction of and A_3
3) In P_2 switch the position of A_2 with the position of D_1
4) In P_4 switch the gluing direction of D_1.

Nonorientable Embedding of $K_{4,2n}$ on X^{n-1}

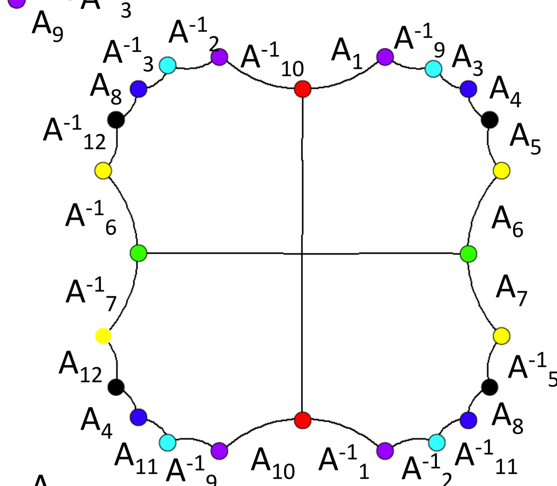
The fundamental domain for $K_{3,4n}$ on X^{2n} consists of three $4n$-gons, P_1, P_2, P_3 with the following edges (listed counterclockwise):

P_1: A_3 then $B_1 B_2 B_3 \ldots B_{4n-4}$ then A_1
P_2: $A_2 A_3$ then $D_1\, B_{4n-4}\, D_2\, B_2\, D_3\, B_{4n-6}\, D_4\, B_4\, D_5\, B_{4n-8} \ldots D_{2n-2}\, B_{2n-2}$
P_3: $B_{2n-1}\, D^{-1}_{2n-3}\, B^{-1}_{2n-3}\, D^{-1}_{2n-2}\, B^{-1}_{2n+1}\, D^{-1}_{2n-5}\, B^{-1}_{2n-5}\, D^{-1}_{2n-4}\, B^{-1}_{2n+3}\, D^{-1}_{n-6}$ $\ldots B^{-1}_{1} D_2$ then $A_1 A_2$.

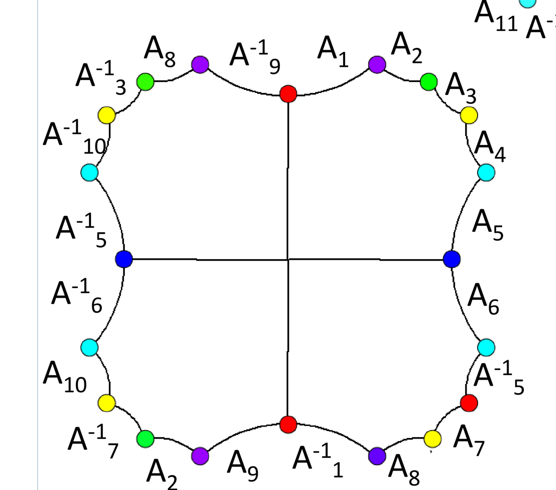
Examples



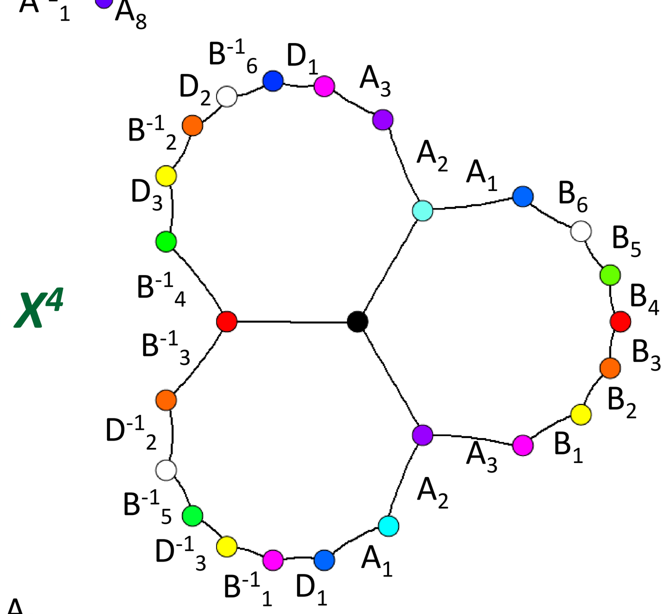
$K_{4,8}$ on O^3



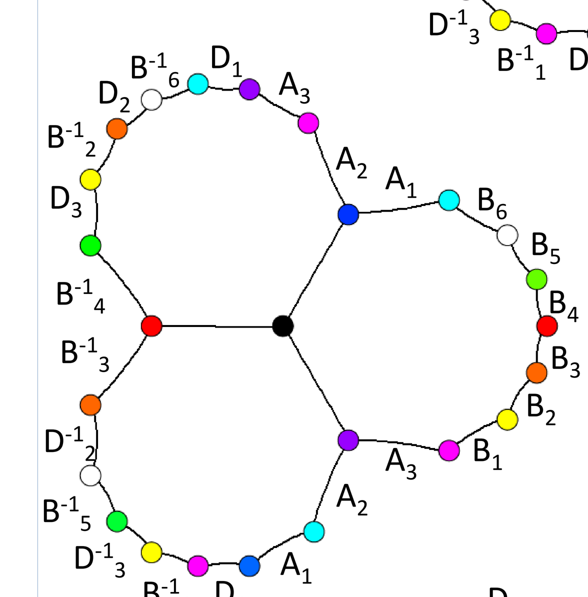
$K_{4,8}$ on X^6



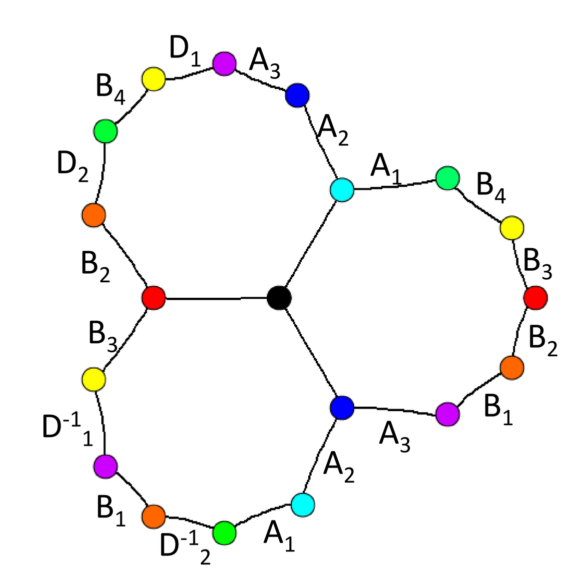
$K_{4,7}$ on X^5



$K_{3,10}$ on X^4



$K_{3,10}$ on O^2

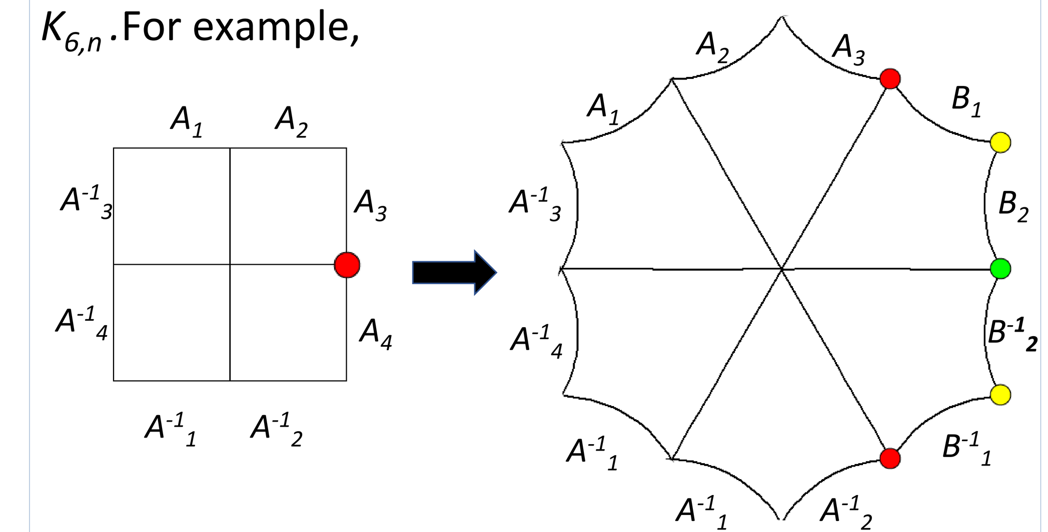


$K_{3,8}$ on X^3

The General Orientable Case

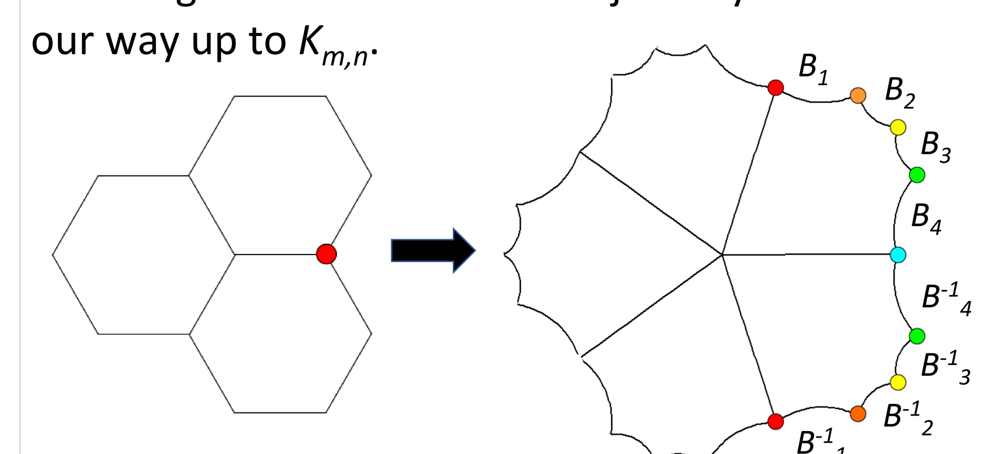
If $(n-2)(m-2)/4$ is an integer than either 2 divides $(n-2)$ and 2 divides $(m-2)$ or 4 divides one of $(n-2)$,(m-2) but not both.

Case 1: $2|(n-2)$ and $2|(m-2)$. Then we know how embed $K_{4,n}$. Next we select a vertex that sits on the edge of two polygons in this embedding. We add two new n-sided polygons, "splitting" the edge and vertex. We assign the two new polygons edge gluings that are mirrored along their common edge. This gives us $K_{6,n}$ .For example,



We repeat this process until we reached have $K_{m,n}$.

Case 2: Without loss of generality we assume that $4|(m-2)$ and that $2\nmid(n-2)$. Then, $m \equiv 2 \bmod 4$, and we know an orientable embedding of $K_{3,m}$. As in the case above, we insert two new polygons, mirroring their edges about the line of adjacency. Then we work our way up to $K_{m,n}$.



The General Nonorientable Case

Luckily, this method works for the nonorientable case as well! If $(m-2)(n-2)/m$ is an integer then either $2|(m-2)$ and $2|(n-2)$ or 2 divides one of $(m-2)$ or $(n-2)$ but not both. In the first case we can work our way up to $K_{m,n}$ from an embedding of the graph $K_{4,n}$ and in the second case we can work out way up from an embedding of the graph $K_{3,n}$. That's all folks!

Acknowledgements

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References

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