For Whom The Bell Tolls: Modeling Wind Chimes with the Classical Wave Equation

Louisa Gao and Matthew Sawicki | Advisor: Dr. Dorothy Wallace

Department of Mathematics, Dartmouth College

Abstract

In solving partial differential equations, we are able to derive solutions to two-dimensional problems from one-dimensional ones. We wonder if the reverse, simplifying a two-dimensional problem to one-dimensional, could be done in solving wave equation. In this project, we model wind chimes of different radii. Intuitively, the wind chimes begin to behave like strings as their radii get smaller, which enables us to model the vibrating surface as a string. In order to test our assumption, we first solve the general wave equation on cylindrical surface and model three wind chimes of various radii against a vibrating string. We then analyze Friture graphs of two wind chimes of different sizes.

Deriving the General Solution to the Wave Equation on the Cylindrical Surface

We consider a wind chime as a cylinder with two open ends. We strike the wind chime in the middle so that the top and bottom experience the same wave. While deriving the general solution to the wave equation, we look at every point on the cylinder in two ways: 1. as a part of a line and 2. as a point on a circular cross-section. We use θ to denote relative position on the circle and x to denote position relative to the bottom of the cylinder. L is the length of the wind chime, and r is the radius of the circular cross-section. Different cylinders have different r's, and the position can be described by rθ, but on the same cylinder, we treat r as a constant. Since x, θ, and t are independent of each other, we can say that u(x, rθ , t) = f(x) · g(rθ)h(t).

General wave equation: $k^2 \bullet u_{tt} = u_{xx} + u_{\theta\theta}$

Boundary conditions: $u(0, r\theta, t) = u(L, r\theta, t)$, $u(x, r\theta, t) = u(x, r(\theta + 2\pi), t)$

Plugging our equation u(x, rθ , t) = f(x)g(rθ)h(t) into the wave equation, we have

$$\frac{f''}{f} + \frac{g''}{g} = \frac{h''}{h}\, k^2 = \lambda = \lambda_1 + \lambda_2$$

To solve for f(x), we use the boundary conditions u(0, rθ, t) = u(L, rθ, t), which translates to f(0) = f(L) and get that

$$f(x) = a \sin\left(\frac{2}{\pi L}x\right) + b \bullet \cos\left(\frac{2}{\pi L}x\right), \text{ where } \lambda_1 = -\frac{4\pi^2}{L^2}$$

To solve for g(rθ), we use u(x, rθ, t) = u(x, r(θ + 2π), t), which translates to g(rθ) = g(r(θ + 2π)). We then have

$$g(r\theta) = c \sin(r\theta) + d \bullet \cos(r\theta), \text{ where } \lambda_2 = -m^2$$

To solve for h(t), we know that $\lambda = \lambda_1 + \lambda_2 = -\frac{4\pi^2}{L^2} - m^2$, so

$$h(t) = u \sin\left(k\left(\frac{2}{\pi L} - m\right)t\right) + v \cos\left(k\left(\frac{2}{\pi L} - m\right)t\right)$$

Thus, the general solution to the wave equation on cylindrical surface is:

$$u(x, r\theta, t) = \sum_{n,m=0}^{\infty} \left[A_n \sin\left(\frac{2}{\pi L}x\right) + B_n \cos\left(\frac{2}{\pi L}x\right)\right]\left[C_n \sin(r\theta) + D_n \cos(r\theta)\right]\left[U_{n,m} \sin\left(k\left(\frac{2}{\pi L} - m\right)t\right) + V_{n,m} \cos\left(k\left(\frac{2}{\pi L} - m\right)t\right)\right]$$

When r = 0, we have:

$$u(x, r\theta, t) = \sum_{n,m=0}^{\infty} \left[A_n \sin\left(\frac{2}{\pi L}x\right) + B_n \cos\left(\frac{2}{\pi L}x\right)\right]\left[U_{n,m} \sin\left(k\left(\frac{2}{\pi L} - m\right)t\right) + V_{n,m} \cos\left(k\left(\frac{2}{\pi L} - m\right)t\right)\right].$$

The general solution to a vibrating string is

$$u(x, t) = \sum_{n,m=0}^{\infty} \left[\sin\left(\frac{n}{\pi L}x\right)\right]\left[A_{n,m} \sin(mt) + B_{n,m} \cos(mt)\right],$$

whose two ends are always fixed. Since our initial condition indicates free ends, we have an additional $\cos\left(\frac{n}{\pi L}x\right)$ term compared to the general one-dimensional solution. We can thus conclude that theoretically, when $r = 0$, our wind chime behaves like a vibrating string.

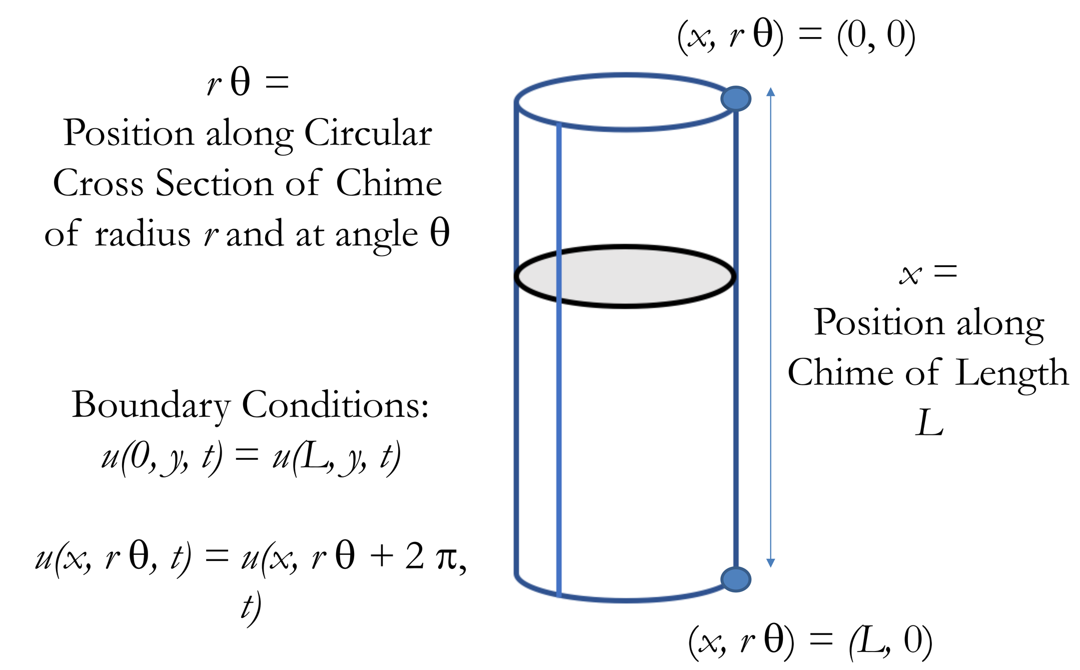
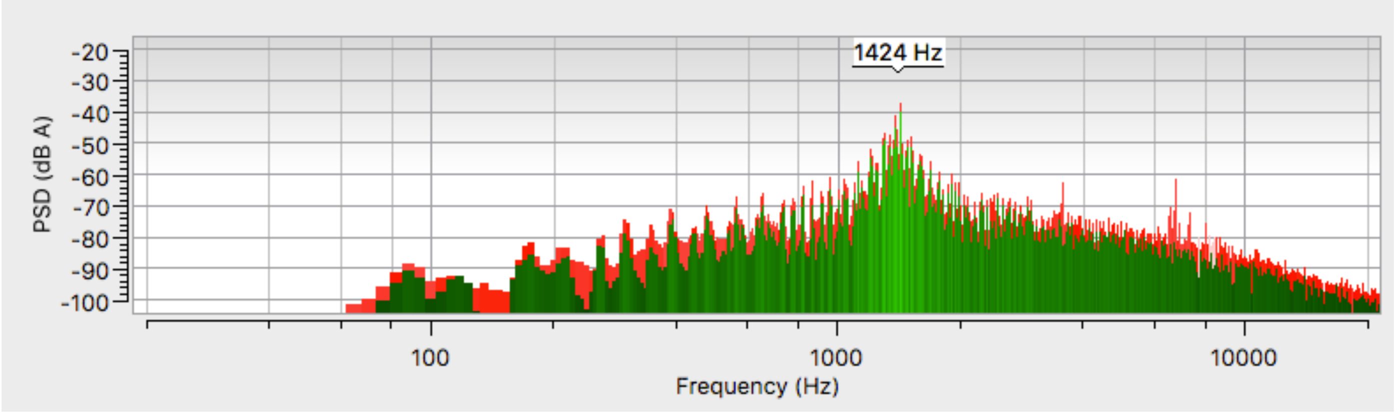


Figure 1. *Visual diagram of the wave equation on a cylindrical surface.* Figure 1 shows a visual diagram of the setup of the classical wave equation on the cylindrical surface. The solution *u(x, r θ, t)* gives the displacement of the vibration of the membrane at a position *x* along the length of the chime and at position *r* θ along the circular cross section of the chime at any given time *t*. The boundary conditions are additionally visually represented in Figure 1.

Empirical Observations of Two Different-Sized Wind Chimes

When a string produces sound, it has one fundamental frequency and many overtones. The frequency of the overtones should be multiples or fractions of the fundamental. In this part, we analyze the behaviors of wind chimes from the frequency sound profile given at the time the chime was struck. Two wind chimes of two different radii and lengths were struck at rest with an approximate same initial velocity at the exact half point of the chime's length. The smaller wind chime had a radius of 0.3 cm and length 15 cm while the larger wind chime had a radius of 2.3 cm and length 31 cm. The sound profiles shown below were taken from the real-time sound analysis software Friture.



Frequency (Hz)	Loudness (dB)	Proportion of Fundamental
1424.00*	-39.144	1
1496.924	-49.955	1.05
1372.777	-43.675	0.96
1581.248	-54.980	1.11
1293.898	-47.958	0.91
1678.031	-57.741	1.17
1764.006	-60.251	1.24

Figure 2 and Table 1. *Sound profile of a cylindrical wind chime with a large radius.* Figure 2 shows output of the noise profile at the exact time a radially-large wind chime was struck. Note the presence of a large number of overtones of the fundamental frequency, 1424 Hz. Table 1 shows data points grabbed from Figure 2 and gives the loudness, in dB, of the loudest overtones produced. Table 1 additionally shows the proportionality of each overtone frequency to the fundamental. The loudest overtones appear to be a factor of $n \cdot 0.05$ away from the fundamental.

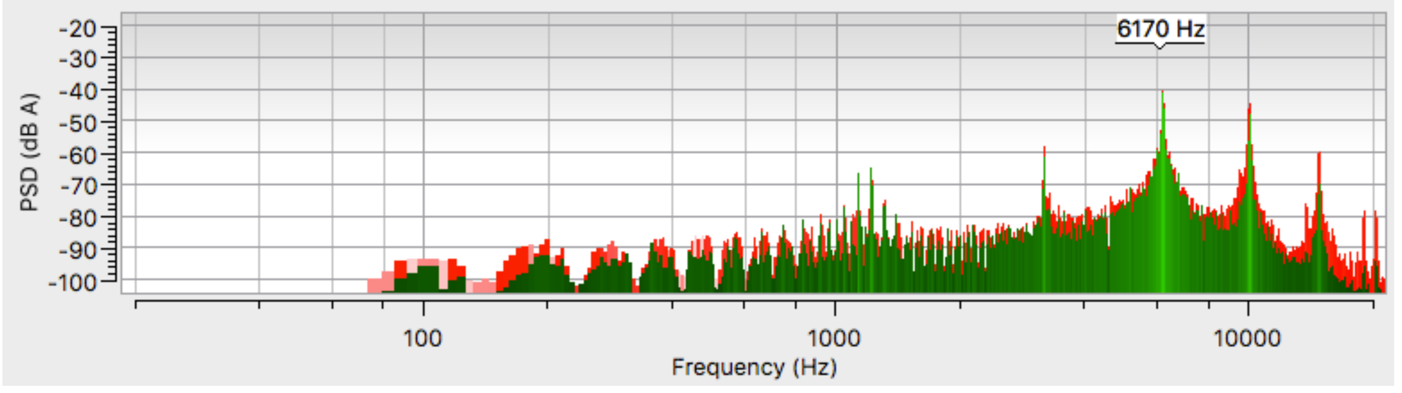


Figure 3 and Table 2. *Sound profile of a cylindrical wind chime with a small radius.* Figure 3 shows output of the noise profile at the exact time a radially-small wind chime was struck. In contrast to the noise profile contained in Figure 2, this profile contains fewer and more distinct overtones, which can be seen at the graph's peaks. Table 2 confirms this interpretation by giving grabbed data points grabbed from Figure 3 and showing the loudness, in dB, of the loudest overtones produced alongside their proportionality to the fundamental frequency, which is 6170.00 Hz. Note that these overtones appear at approximate integer-multiples and integer-fractions of the fundamental, with the exception of 10041.533 Hz. This shows the simplified one-dimensional nature of wind chimes for chimes with smaller radii.

Frequency (Hz)	Loudness (dB)	Proportion of Fundamental
6170.000*	-41.635	1
10041.533	-48.176	1.61
14709.570	-71.320	2.36
18814.817	-95.975	3.01
3159.366	-59.812	0.51
1217.867	-70.817	0.20
1043.758	-76.352	0.17

Analysis of Results

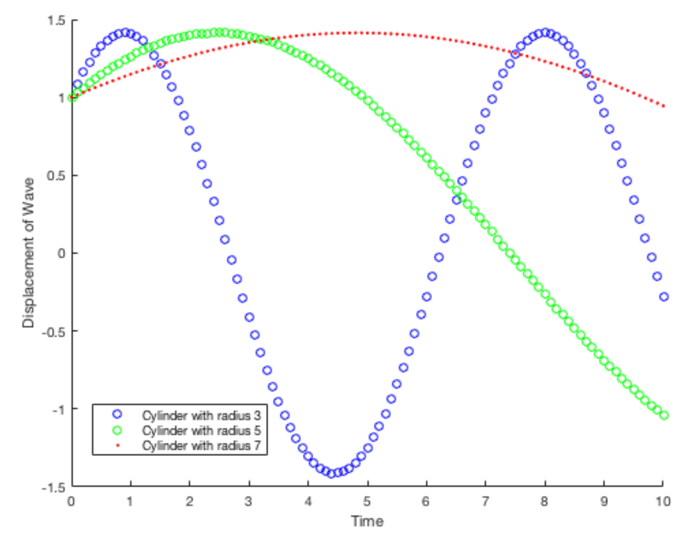


Figure 4. *Effect of changing the cylindrical radius on the shape of the solution u.* Figure 4 shows the graphed solution *u* over time at position *x* = ½*L* and *r* θ = 0 for different values of the radius *r*. All other coefficients were kept the same among each curve. Figure 4 demonstrates how a lower value for *r* produces a curve more similar to that of the wave equation on a string.

Our results are consistent across theoretical and experimental analyses. From the derivation of general solution to the wave equation on cylindrical surface, we are able to conclude that the wind chime behaves like a vibrating string when stroked in the middle and when its radius approaches zero. The graphical analysis of different functions, where we set different radii of the solution previous solved, also show that our modeled vibrating cylindrical membrane approximates a string as its radius decreases. From our comparison of two wind chimes of different radii, we notice that the wind chime of smaller radius demonstrates more frequencies proportional to its fundamental frequency, compared to the wind chime of bigger radius. The spikes shown in the FFT Spectrums are overtones, and the existence of harmonics (multiples and fractions of fundamental frequency) show that the smaller wind chime behaves more like a vibrating string.

Conclusions and Future Directions

Our findings demonstrate that we were able to simplify the vibrating membrane of a cylindrical surface when the radius of the cylinder is relatively small. Thus, in solving the vibration of wind chime, we could approximate a two-dimensional wave equation problem a one-dimensional problem. The boundary conditions on the cylindrical membrane were assumed to be Dirilecht-like, with the displacement of the wave at either end along the length of membrane equal to one another. This assumption was made by reasoning that the strike on the chime occurs halfway along its length, and that the symmetrical nature of waves would then allow for this boundary condition to be satisfied. Further analysis and data collection about chimes should look at solid tubes and determine whether the same kinds of similarities and generalizations can be traced regarding the solution space. Additionally, the study of the impact of different initial conditions on the solution space and the Fourier coefficients should be helpful, i.e. performing strikes on the chime at different positions along its length.

References and Acknowledgements

- Daileda, Ryan C. "The Two Dimensional Wave Equation." http://ramanujan.math.trinity.edu/rdaileda/teach/s12/m3357/lectures/lecture_3_1_short.pdf.
- Mei, C.C. "1.138J/2.062J/18.376J, Wave Propagation, Sample Wave Problems." http://web.mit.edu/1.138j/www/material/chap-1.pdf.
- "The Wave Equation." http://www.damtp.cam.ac.uk/user/dbs26/1BMethods/Wave.pdf
- "Wave Equation." Wolfram MathWorld. Accessed May 27, 2019. http://mathworld.wolfram.com/WaveEquationRectangle.html.
- Yong, Darryl. "An Introduction to Partial Differential Equations in the Undergraduate Curriculum." https://www.math.hmc.edu/~ajb/PCMI/lecture14.pdf.
- We would like to thank Professor Wallace for a great term and Lemon Tree Gifts in Hanover, NH for lending us the wind chimes.