αβγ Conjecture for Gaussian Integers

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The abc Conjecture

The abc conjecture claims that the sum of two numbers that factor a lot will not factor a lot.

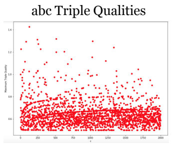
$$a+b=c$$

with $c \geq a, b$ and $\gcd(a, b) = 1$

For ε > 0, there are only finitely many triples with quality > 1+ε where quality, q, of abc is defined as

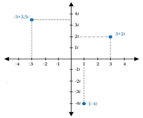
$$q(a,b,c) = \frac{[\log(c)]}{[\log(\text{rad}(abc))]}$$

i.e. $c < \text{rad}(abc)^{1+\varepsilon}$



Background

Gaussian integers are the subring of the complex numbers consisting of elements **α** = a+bi where a,b in **Z**. They are mapped into the complex plane as is shown below:



Definitions

Norm: For **α**= a + bi in **Z**[i], we define its norm by $N(\alpha) = a^2 + b^2 = |\alpha|^2$

Prime Factorization:

Integers: For $n \in \mathbf{Z} \geq 2$ can be written uniquely $n=p_1^{e_1}p_2^{e_2}\ldots p_r^{e_r}$ where $p_1<p_2<\ldots<p_r$ are primes and $e_1,e_2,\ldots,e_r \in \mathbf{Z}\geq 1$

Gaussian integers: Every nonzero **α** in **Z**[i] can be written uniquely as $\alpha = u\pi_1^{e_1}\ldots\pi_r^{e_r}$ where u is a unit, each π_i is a Gaussian prime in the upper right quadrant or the positive real axis, and e_i in $\mathbf{Z}_{\geq 1}$

Radical: For $n \in \mathbf{Z} \geq 2$ and $\alpha \in \mathbf{Z}[i]$

$$\text{rad}(n) := p_1p_2\ldots p_i$$

$$\text{rad}(\pi) := \pi_1\pi_2\ldots\pi_i$$

αβγ Triple: Three nonzero Gaussian integers **α,β,γ** such that

$$\alpha+\beta=\gamma$$

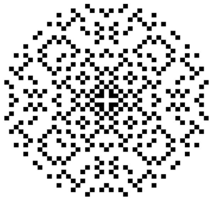
$$N(\gamma) \geq N(\alpha), N(\beta)$$

$$\alpha=a+bi \quad \beta=c+di \quad \gcd(\alpha,\beta) = 1$$

Quality: $q = \log(N(\gamma)) / \log(N(\text{rad}(\alpha\beta\gamma)))$

High Quality Hit: An (**α,β,γ**) triple with quality > 1

Plot of Gaussian Primes:



αβγ Conjecture

Let ε > 0

Then there are only finitely many **αβγ** triples with quality > 1+ε,

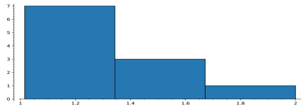
i.e., $N(\gamma) < \text{rad}(N(\alpha\beta\gamma))^{1+\varepsilon}$ for all but finitely many triples

Findings

Hits with High Quality: (-30:30; excluding duplicates)

α	β	γ	q
1	1	2	2
24i+7	1	24i+8	1.652
-24i+7	-24i-7	-48i	1.545
4i+3	1	4i+4	1.505
7i+23	1	7i+24	1.253
3i-4	29i-28	32i-32	1.177
8i+15	1	8i+16	1.123
3i-4	-3i-4	-8	1.063
3i-4	21i-20	24i-24	1.041
7i-24	24i-7	31i-31	1.029
11i-2	-2i+11	9i+9	1.015

Histogram of Quality (-30:30; excluding duplicates):



Implications

Through experiment, we observed an inversely proportional relationship between the number of hits, and the increase in quality of hits over the Gaussian Integers. Thus, it can be noted that the **αβγ** triples for Gaussian integers behave like abc triples for integers. This gives us some experimental confirmation of the **αβγ** conjecture.

References

Gaussian Integers II. (2013, February 27). Retrieved from https://brilliantscholars.wordpress.com/2013/02/26/gaussian-integers-ii/

Silverman, J. H. (2018). A friendly introduction to number theory. New York City, NY: Pearson.

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