Orders in Quaternion and Central Simple Algebras

Sara Chari

Abstract

Orders in quaternion and central simple algebras enjoy a variety of structural properties. We study orders in such algebras in two different contexts: first, we study the metacommutation problem involving a permutation arising from different factorizations of a given element. We then establish the equivalence of two classes of orders, namely Bass and basic orders, in the quaternionic case.

The metacommutation problem is as follows. In a quaternion order of class number one, an element can be factored in multiple ways depending on the order of the factorization of its reduced norm. The fact that multiplication is not commutative causes an element to induce a permutation on the set of primes of a given reduced norm. We discuss this permutation and previously known results about the cycle structure, sign, and number of fixed points for quaternion orders. We generalize these results to other orders in central simple algebras over global fields.

We then study the Bass and basic conditions on quaternion orders. A quaternion order $\mathcal{O}$ over a Dedekind domain R is Bass if every R-superorder is Gorenstein, and $\mathcal{O}$ is basic if it contains an integrally closed (i.e., maximal) quadratic R-order. We first show that these two conditions are equivalent in a local setting. We then show that basic is a local property; i.e. an R-order R is basic if and only if its localizations $\mathcal{O}_{\mathfrak{p}}$ at each prime of R is basic. This allows us to establish the equivalence of Bass and basic orders in the global case.