

The 2-Selmer Group of a Number Field

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Abstract

This thesis provides a detailed study of the 2-Selmer group of a number field. We prove structural results which allow us to make predictions in the style of Cohen-Lenstra for distributions of narrow class groups and unit signatures. We incorporate two theories into heuristics on narrow class groups: genus theory for S_n -number fields of even degree and reflection theorems for abelian number fields of odd degree.

We study the effects of genus theory on 2-torsion in narrow class groups. Given a number field F , this manifests as a quadratic extension of the rationals for which the compositum of the quadratic extension and F is contained in a certain ray class field of F . We classify six Q -imprimitive types of S_n -number fields of even degree — each with a different contribution from genus theory to the 2-torsion in the narrow class group. We make predictions for the proportion of fields with each Q -imprimitive type and for the 2-torsion in the narrow class groups of fields with a fixed Q -imprimitive type.

In joint work with Ila Varma and John Voight we incorporate reflection theorems into heuristics for class groups for abelian number fields of odd degree. We prove results on the Galois module structure of 2-Selmer group of a number field which leads to predictions for unit signature ranks and 2-torsion in ray class groups. In an appendix to the joint paper (with Noam Elkies) we prove that there are infinitely many cyclic cubic fields with a totally positive system of fundamental units.