# Applications of Probabilistic Combinatorics to Markov Chains, Redistricting, and Permutations

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*Abstract*

This thesis concerns several problems arising from probability and combinatorics, with three chapters each dedicated to a topic at the intersection of these two areas. Chapter 1 introduces the concepts of mixing time and Markov chain Monte Carlo, a suite of tools used to approximately sample from probability distributions when it is intractable to sample precisely. We describe a Markov chain whose state space is the set of $\lambda$-weighted monomer-dimer configurations on the $n$-path and $n$-cycle graphs, and use coupling to show that this chain mixes in time $O(n \log n)$, establishing a new upper bound for these graphs.

In Chapter 2, we introduce political redistricting as a graph partitioning problem and review recent applications of Markov chain Monte Carlo to the task of sampling partitions of connected graphs and detecting partisan gerrymandering. We define a Markov chain whose state space is the set of connected partitions of a graph into two parts of fixed sizes and study the problem of when this chain is irreducible, proving that biconnectivity of the underlying graph is necessary and that Hamiltonianicity and near-Hamiltonianicity are sufficient; furthermore we conjecture that the chain is irreducible for all polyhedral graphs. We prove that the chain has exponential mixing time on a family of highly-connected graphs, introduce a weighted variant whose stationary distribution is more heavily concentrated on partitions with short boundaries, and consider applications of the chain to the county graph of Iowa.

In Chapter 3, we study singular permutons: probability measures on the unit square arising as limit objects of permutations. We define tame permutons, a family of permutons supported on a union of simple rectifiable curves subject to some differentiability requirements, and a subfamily of hexagonal permutons. We introduce two flavors of grid classes of permutations, prove that every geometric class is the class of a singular permutation, construct a permuton whose class is the monotone class of the diamond matrix, and conjecture that such a construction is possible in general. We define catropy, a notion of entropy for singular permutons, and establish combinatorial methods for bounding this quantity using permutation griddings. We close by stating some open problems related to catropy.