CATEGORY THEORY AND THE GELFAND-NAIMARK THEOREM

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Contents

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Abstract. We apply category-theoretic techniques alongside the Gelfand-Naimark theorem to show the existence of an adjunction and category equivalences in the field of functional analysis. After introducing the category of Banach spaces and the category of topological spaces, we present two functors between these categories and an adjunction between the functors. From this adjunction, we can produce the Gelfand transform; because the transform is idempotent, we show that there is an equivalence of categories between two subcategories of our original categories, and we argue that one of the subcategories can be identified by the category of compact Hausdorff spaces. After explaining the notion of a C*-algebra, we discuss how the Gelfand-Naimark theorem identifies the category of C*-algebras with the second of the subcategories.

1. Introduction

To the reader without a sturdy mathematical background, even a very loose explanation of the Gelfand-Naimark theorem is almost meaningless: conveying the power of this theorem in layman's terms is a real challenge. If the vague description of the theorem is difficult enough to comprehend, the precise technical statement and proof of the Gelfand-Naimark theorem are harder still; in order to prove the theorem, one must draw from as far afield as the Stone-Weierstrass theorem and spectral theory. And after wading through the arguments, one comes to the realization that the proof does not reveal much about the deep mathematical structures at work in the theorem. This thesis aims to provide precisely that understanding. By using a wholly different branch of mathematics — namely, category theory — we arrive at a more intuitive conceptualization of the theorem and its mathematical importance.

In 1943, Israel Gelfand and Mark Naimark published the Gelfand-Naimark theorem; although their work would eventually serve as a central result in the study of C*-algebras, their original paper (see Gelfand and Naimark [2]) does not even introduce the formal notion of a C*-algebra. Their theorem asserts that certain varieties of C*-algebras can be considered almost a relabelling of the object obtained once one subjects the C*-algebra to the Gelfand transform.

In this paper, we show how to employ the power of category theory to reorganize the proof of the theorem and derive novel results in this context. In particular, we consider structures and transformations invoked in the proof of the Gelfand-Naimark theorem as examples of elementary concepts in category theory. Once we revisit the Gelfand-Naimark theorem within the context of category theory, then, we can analyze surprising relationships that fall out from the theory quite intuitively. Only small bits of the proof of the theorem can be omitted entirely — the Gelfand-Naimark theorem is a deep statement and requires a deep proof. Reframing the theorem as a statement in a category-theoretic context, however, does permit us to understand exactly where these difficult results are necessary, and we can produce results that serve as important examples in category theory. Moreover, category theory gives us the ability to expand exactly the impact or utility of the Gelfand-Naimark theorem; instead of viewing it as an abstruse result specific to C*-algebras and functional analysis, we can see the theorem as providing the base for a deep relationship between the category of C*-algebras and the category of compact Hausdorff spaces.

In chapter 2, we illustrate how, with very little mathematical architecture, one can simply define two categories — the category of Banach algebras and the category of topological spaces. Chapter 3 presents two functors connecting these categories. In chapter 4, we illustrate that there is an adjunction of these functors, and in chapter 5, we argue that the classical Gelfand transform arises naturally from this adjunction. Chapter 6 introduces the concept of an equivalence of categories to argue that two subcategories of our original categories are equivalent, and that the category of compact Hausdorff spaces is equivalent to one of the subcategories. Chapter 7 discusses C*-algebras, and we also present the argument that the category of C*-algebras is equivalent to the second subcategory. Crucially, this proof will invoke the Gelfand-Naimark theorem. We prove the theorem itself in chapter 8.

2. Two Categories

2.1. **Categories.** Throughout this paper, we use only the precise notions from category theory that we need for our purposes. For a fuller treatment, see Maclane [4]. First we recall the following definition.

Definition 2.1. A **category** C requires the following:

(1) A class of objects denoted Obj_C.
(2) For every pair of objects X, Y in Obj_C, there exists a set of morphisms $Hom_C(X, Y)$, where each element $f \in Hom_C(X, Y)$ defines a morphism or "arrow" $f : X \to Y$.
(3) For every three objects X, Y, Z in Obj_C, there is a binary operation

$$\circ : Hom_C(Y, Z) \times Hom_C(X, Y) \to Hom_C(X, Z)$$

which satisfies:

Associativity: . For each morphism f, g, h where $f : W \to X$, $g : X \to Y$, $h : Y \to Z$, $(h \circ g) \circ f = h \circ (g \circ f)$.

Identity: For each object X there exists $I_X \in Hom_C(X, X)$ such that for each $f \in Hom_C(X, Y)$, $f \circ I_X = f$ and for each $f \in Hom(Y, X)$, $I_X \circ f = f$.

Observe that Obj_C is not necessarily a set; for example, it is well-known that the collection of all groups is not a set.

Remark. In this paper, we consider categories whose morphisms are functions, so associativity is immediate, as compositions of functions are associative. Moreover, the categories we consider have obvious identity maps that are functions from elements to themselves. In our early examples, we need to check that composition of morphisms yields a morphism in the appropriate category — that is, that the composition operation is closed for the functions we consider, and that the identity maps are actually morphisms.

2.2. **The Category $\mathcal{B}$an.** Now, we build up to the definition of a Banach algebra. Recall the following elementary definitions.

Definition 2.2. A norm $\|\cdot\|$ on a complex vector space X is a function $\|\cdot\| : X \to \mathbb{R}$ which satisfies the following properties:

(1) $\|x\| \geq 0$ for all $x \in X$.
(2) $\|x\| = 0 \iff x = 0$.
(3) $\|\lambda x\| = |\lambda|\|x\|$ for $\lambda \in \mathbb{C}$ and $x \in X$.
(4) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$ (Triangle Inequality).

Definition 2.3. A metric space X is **complete** if every Cauchy sequence in X converges to a point in X.

Remark. If a normed space X is imbued with a norm $\|\cdot\|$, define a distance d between points $x, y \in X$ as $d(x, y) = \|x - y\|$. Observe that $d(x, y) \geq 0$. Also note that $d(x, y) = 0 \iff \|x - y\| = 0 \iff x = y$, and that for any $z \in X$, $d(x, z) = \|x + (-y + y) - z)\| \leq \|x - y\| + \|y - z\| = d(x, y) + d(y, z)$, so we have the metric version of the triangle inequality. As a result, every normed vector space gives a metric space.

Definition 2.4. A complete, normed vector space is called a **Banach space**.

Definition 2.5. An **associative algebra** is a vector space A equipped with the bilinear associative product $\cdot : A \times A \to A$ where $\cdot$ satisfies:

(1) $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, for all $x, y, z \in A$.
(2) $x \cdot (y + z) = x \cdot y + x \cdot z$, for all $x, y, z \in A$.
(3) $(x + y) \cdot z = x \cdot z + y \cdot z$, for all $x, y, z \in A$.

(4) $\lambda(x \cdot y) = (\lambda x) \cdot y = x \cdot (\lambda y)$ for $x, y \in A$ and $\lambda \in \mathbb{C}$, if $\mathbb{C}$ is the underlying field for A.

Note that there exist non-associative algebras which do not satisfy (1). We only consider associative algebras in this paper. To abbreviate the discussion, when we describe an "algebra," we mean associative algebras.

Definition 2.6. A norm is **submultiplicative** on an algebra A if for all $a, b \in A$, $\|ab\| \leq \|a\|\|b\|$.

Naturally, that gives us:

Definition 2.7. A **Banach algebra** is an algebra that has a complete submultiplicative norm.

Finally, we need to refine our understanding of linear operators on a vector space.

Definition 2.8. Consider a linear operator $f : W \to V$, where W and V are two Banach spaces. f is a **bounded** linear operator if there exists $M > 0$ such that $\|f(x)\|_V \leq M\|x\|_W$ for all $x \in X$.

Remark. We define $\|f\| = \inf\{M \in \mathbb{R} : \|f(x)\|_V \leq M\|x\|_W \text{ for all } x \in X\}$. It is a fact in functional analysis that that this norm, the operator norm, is indeed a norm.

Proposition 2.1. *The objects that are unital abelian Banach algebras whose underlying field is $\mathbb{C}$, alongside the morphisms that are bounded unital algebra homomorphisms between the unital abelian Banach algebras, form a category denoted $\mathcal{B}$**an**.*

Consider arbitrary unital abelian Banach algebras A, B, and C and bounded unital algebra homomorphisms f, g, h where $f : A \to B$, $g : B \to C$, and $h : C \to D$. First we claim that composition of bounded unital algebra homomorphisms yields a unital algebra homomorphism; that is, $g \circ f \in Hom_{\mathcal{B}\mathbf{an}}(A, C)$. For $a, b \in A$, we have $f(a \cdot b) = f(a) \cdot f(b)$, so $g(f(a \cdot b)) = g(f(a) \cdot f(b)) = g(f(a)) \cdot g(f(b))$; as a result, $(g \circ f)(a \cdot b) = (g \circ f)(a) \cdot (g \circ f)(b)$, and similarly for addition, so $g \circ f$ is an algebra homomorphism. Denote 1_G as the unit in arbitrary unital Banach algebra G. Then, $f(1_A) = 1_B$, since f is a unital algebra homomorphism. By the same argument, $g(1_B) = 1_C$. So $g(f(1_A)) = (g \circ f)(1_A) = 1_C$. So we have that $g \circ f$ is a unital algebra homomorphism from A to C which therefore lies in $Hom_{\mathcal{B}\mathbf{an}}(A, C)$.

Now, we need to show $g \circ f$ is bounded as a linear operator. Because f and g are bounded,

$$\|g(b)\|_C \leq M \cdot \|b\|_B, \text{ for all } b \in B,$$

and

$$\|f(a)\|_B \leq N \cdot \|a\|_A, \text{ for all } a \in A.$$

In that case,

$$\|(g \circ f)(a)\|_C \leq M\|f(a)\|_B \leq M \cdot N \cdot \|a\|_A,$$

so $g \circ f$ is bounded.

Second, we claim homomorphisms are associative, i.e., $(h \circ g) \circ f = h \circ (g \circ f)$. This is immediate because homomorphisms are functions, and because composition of functions is associative.

Third we claim that we can induce an identity homomorphism on any Banach algebra as follows. For an arbitrary unital abelian Banach algebra A, construct $I_A(a) = a$ for all $a \in A$. Note that $I_A \in Hom_{\mathcal{B}\mathbf{an}}(A, A)$. For example, $I_A(a \cdot b) = a \cdot b = I_A(a) \cdot I_A(b)$ for $a, b \in A$; and $I_A(1_A) = 1_A$. Now if we take unital abelian Banach algebra B, $f : A \to B$, and $g : B \to A$, we have $f \circ I_A = f$ and $I_A \circ g = g$. Therefore $\mathcal{B}\mathbf{an}$ is a category.

2.3. **The Category $\mathcal{T}$op.** Familiarity with a few topological facts is required to read this paper. We jog the reader's memory of the most important properties invoked here. Let us note the definition of a topological space:

Definition 2.9. A topological space is a double $\langle X, \tau \rangle$ where X is a set and $\tau \subset P(X)$, where $P(X)$ is the power set of X, such that:

1. $\emptyset \in \tau$ and $X \in \tau$.
2. The (finite or infinite) union of elements of τ is also contained in τ.
3. The finite intersection of elements of τ is also contained in τ.

The elements of τ are declared to be open sets in X.

Definition 2.10. Let $f : D \to D'$, where D and D' are both metric spaces imbued with metrics d and d', be a function. f is **continuous** at z_0 if the following condition holds: For every $\varepsilon > 0$, there exists $\delta > 0$, such that for all $z \in P$,

$$d(z, z_0) < \delta \implies d'(f(z), f(z_0)) < \varepsilon.$$

In some cases, we wish to consider continuous functions without considering metrics or norms. Then the following definition is equivalent.

Definition 2.11. Let $f : A \to A'$, where A and A' are both topological spaces, be a function. If for every open set $B' \subset A'$, the inverse image $f^{-1}(B')$ yields an open set in A, then f is **continuous.**

Proposition 2.2. *Let A, B, C be topological spaces, and let $f : A \to B$, $g : B \to C$ be continuous maps. Then $g \circ f : A \to C$ is continuous.*

Proof. Consider any open set $D \subset C$. $g^{-1}(D)$ is open, because g is continuous. $(f^{-1} \circ g^{-1})(D)$ is also open, because f is continuous. So for any open set in C the inverse image $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})(D)$ is also open. Hence $g \circ f$ is continuous. □

Proposition 2.3. *The objects that are topological spaces, alongside the morphisms that are continuous functions from topological spaces to topological spaces, form a category denoted* $\mathcal{T}\mathbf{op}$.

Proof. Consider topological spaces W, X, Y, Z and continuous functions $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$. First, from the above proposition, we know that $g \circ f : X \to Z$ is a continuous map which lies in $Hom_{\mathcal{T}\mathbf{op}}(X, Z)$. The composition of continuous functions yields a continuous morphism, and we immediately have that $(h \circ g) \circ f = h \circ (g \circ f)$. Finally, every topological space is imbued with an identity arrow. For any topological space X and for any $x \in X$, we can construct the function $I_X \in Hom_{\mathcal{T}\mathbf{op}}(X, X), I_X(x) = x$. Observe that $I(x)$ is continuous, because the inverse image of any open set will be mapped to itself, which is open. Consider Y, another topological space. Now for any arrows $f : X \to Y$ and $g : Y \to X$, $f \circ I_X = f$ and $I_X \circ g = g$. □

3. Two Functors

3.1. Functors.

Categories can be linked by functors in the following way.

Definition 3.1. A **covariant functor** F between categories C and D assigns an object $F(X)$ in D to every object X in Obj_C. Moreover, for every X, Y in Obj_C, and every morphism $f \in Hom_C(X, Y)$, F assigns a morphism $F(f) \in Hom_D(F(X), F(Y))$ such that:

- **Identity-preserving:** For each X in Obj_C, $F(I_X) = I_{F(X)}$.
- **Composition-preserving:** $F(g \circ f) = F(g) \circ F(f)$ for $f : X \to Y$ and $g : Y \to Z$.

Definition 3.2. A **contravariant functor** F between categories C and D assigns an object $F(X)$ to every object X in Obj_C. Moreover, for every X, Y in Obj_C, and every morphism $f \in Hom_C(X, Y)$, F assigns a morphism $F(f) \in Hom_D(F(Y), F(X))$ (reverses arrows) such that:

- **Identity-preserving:** For each X in Obj_C, $F(I_X) = I_{F(X)}$.
- **Composition-reversing:** $F(g \circ f) = F(f) \circ F(g)$ for $f : X \to Y$ and $g : Y \to Z$.

3.2. The Functor M.

3.2.1. *$M(A)$ as a map of objects.*

Definition 3.3. For A, a unital abelian Banach algebra, define the topological space $M(A)$ as the set $Hom_{\mathcal{B}\mathbf{an}}(A, \mathbb{C})$ with the topology that it inherits as a subspace of A^*, the dual space of A.

This definition requires some comment. We introduce the weak-* topology as follows.

Definition 3.4. Let W be a Banach space, let W^* be its dual space, and let $f \in W^*$. The weak-* topology on W^* is the weakest topology such that the evaluation map

$$ev_w : W^* \to \mathbb{C}$$
$$ev_w(f) = f(w)$$

is continuous for every $w \in W$.

Remark. Fix $w \in W$. Then, $ev_w : f \mapsto f(w)$ is continuous if for every open $U \in \mathbb{C}$, the inverse image $ev_w^{-1}(U)$ is open in W^*. Therefore, this definition is equivalent to declaring the following topological basis:

$$V_{d,U}^W = \{\varphi \in W^* : \varphi(d) \in U\}.$$

We take as given that this basis can be used to generate a topology.

Observe that $M(A)$ is a subset of A^*, the dual space of A. Therefore $M(A)$ is a topological space under the subspace topology induced by the weak-$*$ topology on A^*.

3.2.2. *M as a map of morphisms.* Now, we consider a map of morphisms M. For any unital algebra homomorphism $f : A \to B$, and for $\varphi \in M(B)$, construct

$$M(f) : M(B) \to M(A)$$
$$M(f)(\varphi) = \varphi \circ f$$

Note that M is indeed well-defined, as $f : A \to B$, $f \in Hom_{\mathcal{B}\mathbf{an}}(A, B)$ while $\varphi : B \to \mathbb{C}, \varphi \in Hom_{\mathcal{B}\mathbf{an}}(B, \mathbb{C})$. So the composition $\varphi \circ f \in Hom_{\mathcal{B}\mathbf{an}}(A, \mathbb{C})$, since $\mathcal{B}\mathbf{an}$ is a category. As a result, we have shown that $\varphi \circ f \in M(A)$.

Proposition 3.1. *For any algebra homomorphism f between unital abelian Banach algebras A and B, the map $M(f) : M(B) \to M(A)$ is continuous.*

Proof. Any element in $M(B)$ is a multiplicative linear functional $\varphi : B \to \mathbb{C}$. Consider any continuous map $f : A \to B$, two Banach* algebras.

Now, we claim that for *any* A^*, B^* — the duals of these Banach* algebras — the map

$$\tilde{f} : B^* \to A^*$$
$$\varphi \mapsto \varphi \circ f$$

is continuous through the weak-$*$ topology. Consider an open set V in A^*. We aim to show the inverse image

$$\tilde{f}^{-1}(V) = \{\varphi \in B^* : \varphi \circ f \in V\}$$

yields an open set. Recall the following topological fact:

Fact. Let X and Y be topological spaces, and let $f : X \to Y$ be a map. If $f^{-1}(D)$ is open in X for all D in a topological basis of Y, then f is a continuous map. (See Munkres [5], section 2.7.)

Now, because we employ a weak-$*$ topology on A^*, it suffices to prove that $\tilde{f}^{-1}(V)$ is open for $V = V^A_{a,U} = \{\psi \in A^* : \psi(a) \in U\}$ for some $a \in A, U \subset \mathbb{C}$. So if $\tilde{f}(\varphi) = \varphi \circ f \in V$, then $(\varphi \circ f)(a) \in U$. Hence we have:

$$\begin{aligned} \tilde{f}^{-1}(V^A_{a,U}) &= \{\varphi \in B^* : (\varphi \circ f)(a) \in U\} \\ &= \{\varphi \in B^* : \varphi(f(a)) \in U\} \\ &= V^B_{f(a),U}. \end{aligned}$$

Since we likewise employ the weak-$*$ topology on B^*, we find that $\tilde{f}(V^A_{a,U})$ yields $V^B_{f(a),U}$, which was declared an open set in B* under this topology. Hence for any open set in A^*, the inverse image of $\tilde{f}$ yields an open set in B^*. Hence $\tilde{f}$ is continuous.

Now, $M(A) \subset A^*, M(B) \subset B^*$. So since the continuity of $\tilde{f}$ holds for general duals, it certainly holds once we restrict the domain to $M(B)$, and hence $M(f)$ is continuous. □

3.2.3. *Proving M is a functor.* We assembled the architecture in the previous sections in order to show the following:

Theorem 3.2. *M is a functor from $\mathcal{B}$**an** to $\mathcal{T}$**op**.*

Proof. $M(A)$ is a topological space under the weak-$*$ topology from A^*, the dual space of $M(A)$. Therefore any $M(A)$ is in $Obj_{\mathcal{T}\mathbf{op}}$ for any unital abelian Banach space A. Moreover, from Proposition 3.1, we know that $M(f)$ is in $Hom_{\mathcal{T}\mathbf{op}}$ for any function f.

Now we verify that $M(I_A) = I_{M(A)}$. For $\varphi \in M(A)$,

$$M(I_A)(\varphi) = \varphi \circ I_A = \varphi.$$

So $M(I_A)(\varphi) = I_{M(A)}$.

Second we verify that

$$\begin{aligned} &M(g \circ f) : M(C) \to M(A) \\ &M(g \circ f) = M(f) \circ M(g). \end{aligned}$$

For $\psi \in M(C)$,

$$\begin{aligned}
[M(f) \circ M(g)](\psi) &= M(f)\big((M(g)(\psi))\big) \\
&= M(f)((\psi \circ g)) \\
&= \psi \circ g \circ f \\
&= \psi \circ (g \circ f) \\
&= M(g \circ f)(\psi)
\end{aligned}$$

□

Therefore, we produce the following:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g \circ f"] & B \arrow[d, "g"] \arrow[rr, bend left, "\mathbf{M}"] & & M(A) & (B) \arrow[l, "M(f)"] \\
& C & & & M(C) \arrow[u, "M(g)"] \arrow[ul, "M(f) \circ M(g)"]
\end{tikzcd}$$

3.3. **The Functor C_b.**

3.3.1. *C_b as a map of objects.* Let X be any topological space. Then let $C_b(X)$ be the set of bounded, continuous functions on X, i.e., the set of continuous functions $f : X \to \mathbb{C}$ such that there exists $M > 0$ with $|f(x)| < M$ for all $x \in X$.

Theorem 3.3. *Let X be a topological space. Then $C_b(X)$ is a unital abelian Banach algebra.*

Proof. First, addition and multiplication of functions are certainly commutative and associative operations, and there exists additive inverses for all continuous functions (namely, taking the additive inverse of the function in $\mathbb{C}$); the map $0(x) = 0$ for all X serves as the additive identity, while the map $1(x) = 1 \in \mathbb{C}$ serves as the multiplicative identity. The distributive property holds with multiplication and addition of functions, and scalars in $\mathbb{C}$ can be multiplied to functions in any order. Finally, we argue that $C_b(X)$ is closed under $+$ and $\cdot$. Take f, g such that $|f(x)| < M$ and $|g(x)| < N$ for all $x \in X$. Then $|(f + g)(x)| < M + N$ and $|(f \cdot g)(x)| < M \cdot N$, and both $M + N$ and $M \cdot N$ are finite integers, so we have that $C_b(X)$ is a unital abelian algebra.

Now, we need to show that $C_b(X)$ is normed. Equip $C_b(X)$ with the supremum norm:

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}.$$

$\|f\|_\infty$ is well defined for all f because $|f(x)| < M$ for all $x \in X$. Moreover, $\|f\|_\infty$ is a norm on $C(X)$. The only nontrivial condition to check is the triangle inequality, as the supremum is linear in scalars and the norm on $\mathbb{R}$ is a norm. Consider $f, g \in C_b(X)$. Then

$$\begin{aligned}
\|f+g\|_\infty = \sup\{|(f+g)(x)| : x \in X\} &= \sup\{|f(x)+g(x)| : x \in X\} \\
&\leq \sup\{|f(x)| + |g(x)| : x \in X\} \\
&= \sup\{|f(x)| : x \in X\} + \sup\{|g(x)| : x \in X\} \\
&= \|f\|_\infty + \|g\|_\infty.
\end{aligned}$$

Moreover, this norm is submultiplicative by the exact same argument, replacing $+$ with $\cdot$.

We claim that $C_b(X)$ is complete. First, take a Cauchy sequence

$$(f_n)_{n\in N} \subset C_b(X),$$

and recall that $\mathbb{C}$ is complete. Fix $x \in X$ to obtain $(f_n(x))_{n\in N}$. Because

$$\|f_n - f_m\|_\infty \geq |f_n(x) - f_m(x)|,$$

it follows that if $(f_n)_{n\in N}$ is Cauchy, then $(f_n(x))_{n\to\infty}$ is certainly Cauchy. As a result, the limit

$$\lim_{n\to\infty} f_n(x) = f(x)$$

exists, and we obtain a function $f : X \to \mathbb{C}$.

It remains to be shown that $f \in C_b(X)$ and that f_n converges to $f \in C_b(X)$. In order to do so, we will need to argue that for any ε, there exists an N such that for $n > N$, $\|f_n - f\|_\infty < \varepsilon$. Fix $\varepsilon > 0$. Then because $(f_n)_{n\in N}$ is Cauchy, there exists $N > 0$ such that for $m, n > N$,

$$\|f_n - f_m\|_\infty < \frac{\varepsilon}{2},$$

Choose x. Now, because $f(x) = \lim_{n\to\infty} f_n(x)$, choose $m > N$ such that

$$|f_m(x) - f(x)| < \frac{\varepsilon}{2}.$$

Now,

$$|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)| < \varepsilon.$$

But for every x, we can choose an $m > N$ which satisfies the above. Hence the choice of N does not depend on x. As a result, we know that

$$\|f_n - f\|_\infty = \sup\{|f_n(x) - f(x)|, \text{ for all } x \in X\} < \varepsilon.$$

First, we claim that f is bounded. By the prior argument, there is certainly some $N > 0$ such that for all $n > N$,

$$\|f_n - f\|_\infty < 1.$$

Invoking the triangle inequality, we see that $\|f\|_\infty \leq \|f_n\|_\infty + 1 < M_1 + 1 = M_2 < \infty$. Therefore so long as each f_n is bounded, f must also be bounded.

Now we also check that f is continuous. Fix $\varepsilon > 0$ and fix $x \in X$. As we have shown, there exists $N > 0$ such that for $m, n > N$,

$$\|f_n - f\|_\infty < \frac{\varepsilon}{3}, \text{ as well as } \|f_n - f_m\|_\infty < \frac{\varepsilon}{3}.$$

As f_n is continuous at x, there exists some open $U \subset X$ with $x \in X$ such that

$$y \in U \implies |f_n(x) - f_n(y)| < \frac{\varepsilon}{3}.$$

Recall that for any $x, y \in X$,

$$|f(x) - f_n(x)| \leq \|f_n - f\|_\infty \text{ and } |f_m(y) - f(y)| \leq \|f - f_m\|_\infty.$$

As a result, we have that there exists some open $U \subset X$ such that for $m, n > N$,

$$y \in U \implies |f(x) - f(y)| \leq |f(x) - f_n(x)| + |f_n(x) - f_m(y)| + |f_m(y) - f(y)| < \varepsilon.$$

So f is continuous in X.

Now, f is bounded and continuous, so $f \in C_b(X)$. As there exists an N such that $\|f_n - f\|_\infty > \varepsilon$ for all $n > N$, we have therefore shown that $\lim_{n\to\infty} f_n = f$.

As a result, we have that any Cauchy sequence in $C_b(X)$ converges to a limit point in $C_b(X)$, so $C_b(X)$ is complete for any topological space X. Therefore $C_b(X)$ is complete, and we have shown that $C_b(X)$ is a unital abelian Banach algebra. $\square$

3.3.2. *C_b as a map of morphisms.* Now consider all continuous functions $f : X \to Y$ for X, Y which are topological spaces, i.e., $f \in Hom_{\mathcal{T}\mathbf{op}}(X, Y)$. Also take bounded continuous functions $g \in C_b(Y)$. Then define a map of morphisms.

$$C_b(f) : C_b(Y) \to C_b(X),$$
$$(C_b(f))(g) = g \circ f.$$

Observe that C_b is well-defined. If $f : X \to Y$ is a continuous function, and $g : Y \to Z$ is a bounded, continuous function, then $g \circ f$ is also continuous at each $x_0 \in X$, since $f \in Hom_{\mathcal{T}\mathbf{op}}(X, Y)$ and $g \in Hom_{\mathcal{T}\mathbf{op}}(Y, \mathbb{C})$. As g is bounded, $g(f(x))$ is certainly bounded. As a result, $C_b(f)$ accepts elements in $C_b(Y)$ and returns elements in $C_b(X)$.

Proposition 3.4. *Let $f : X \to Y$ be a continuous map on topological spaces X, Y. $C_b(f)$ is a bounded unital algebra homomorphism.*

Proof. $C_b(f)$ is clearly an algebra homomorphism. Observe,

$$C_b(f)(z_1 + z_2) = (z_1 + z_2) \circ f = z_1 \circ f + z_2 \circ f = C_b(f)(z_1) + C_b(f)(z_2).$$

The same holds true with multiplication. Now, for $1_{C_b(Y)}$, the multiplicative unit in $C_b(Y)$ which sends $x \mapsto 1 \in \mathbb{C}$, $C_b(f)(1_{C_bY}) = 1_{C_bY} \circ f = 1_{C_bX} \in C_b(X)$, so $C_b(f)$ preserves the unit. As a result, $C_b(f)$ is a unital algebra homomorphism.

Finally, we need to show that $C_b(f)$ is bounded as a map of Banach algebras. Observe that

$$\sup\{|g(f(x))|, \text{ for all } x \in X\} \leq \sup\{|g(y)|, \text{ for all } y \in Y\},$$

as $\{f(x), \text{ for all } x \in X\} \subseteq \{y \in Y\}$.

As a result, we have that

$$\|g \circ f\|_\infty \leq \|g\|_\infty.$$

But this means

$$\|C_b(f)(g)\|_\infty \leq 1 \cdot \|g\|_\infty,$$

so $C_b(f)$ is a bounded linear operator of norm 1. □

3.4. **Proving C_b Is a Functor.**

Theorem 3.5. *C_b is a contravariant functor from $\mathcal{T}$**op** to $\mathcal{B}$**an**.*

Proof. For any X that is in $Obj_{\mathcal{T}\mathbf{op}}$, $C_b(X)$ is in $Obj_{\mathcal{B}\mathbf{an}}$, because it is a Banach algebra. For any $f \in Hom_{\mathcal{T}\mathbf{op}}$, $C_b(f) \in Hom_{\mathcal{B}\mathbf{an}}$, since it is a bounded unital algebra homomorphism.

Now, we check that C_b satisfies the properties of functors. First, for any $f \in C_b(Y)$,

$$C_b(I_Y)(f) = f \circ I_Y = f \in C_b(Y).$$

So C_b preserves the identity map, because $C_b(I_Y) = I_{C_b(Y)}$.

Next we check that $C_b(g \circ f) = C_b(f) \circ C_b(g)$. Consider arbitrary $g : Y \to Z$ and $f : X \to Y$. $C_b(f)$ accepts any function $z_1 \in C_b(Y)$ and composes them to generate $z_1 \circ f : C_b(Y) \to C_b(X)$. $C_b(g)$ accepts any function $z_2 \in C_b(Z)$ and composes them to generate $z_2 \circ g : C_b(Z) \to C_b(Y)$. Therefore for any $z_2 \in C_b(Z)$,

$$[C_b(f) \circ C_b(g)](z_2) = (C_b(f))(z_2 \circ g) = z_2 \circ g \circ f$$

by compositions of functions. This is equivalent to $C_b(g \circ f)(z_2) = z_2 \circ (g \circ f)$ and we have therefore shown C_b is a contravariant functor.

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "g \circ f"'] & Y \arrow[d, "g"] \arrow[rr, bend left, "\mathbf{C_b}"] & & C_b(X) & C_b(Y) \arrow[l, "C_b(f)"] \\
& Z & & & C_b(Z) \arrow[u, "C_b(g)"] \arrow[ul, "C_b(f) \circ C_b(g)"]
\end{tikzcd}
$$

$\square$

4. Adjoint Functors

We can begin to harness the real power of category theory once we identify that M and F are something called an adjunction.

Definition 4.1. Consider categories C, D. Then an **adjunction** from C to D is a triple: $\langle F, G, \Phi \rangle$ where $F : C \to D$ and $G : D \to C$ are functors, and $\Phi : Hom_C(FX, A) \to Hom_D(X, GA)$ is a bijection of sets which is *natural* in the following sense:

Let X, Y be objects in Obj_C and A, B be objects in Obj_D. Then for every morphism $f \in Hom_C(X, GA)$, $g \in Hom_D(FX, A)$, $h \in Hom_C(Y, X)$, and $k \in Hom_D(A, B)$, we have:

(1) $\Phi(G(k) \circ f) = k \circ \Phi(f)$.
(2) $\Phi^{-1}(g \circ Fh) = \Phi^{-1}(g) \circ h$.

That is, we obtain the following diagrams for an adjunction:

$$
\begin{tikzcd}
X \arrow[r, "f"] \arrow[dr, "Gk \circ f"'] & GA \arrow[d, "Gk"] \arrow[rr, "\mathbf{\Phi}"] & & FX \arrow[r, "\Phi(f)"] \arrow[dr, "k \circ \Phi(f)"'] & A \arrow[d, "k"] \\
& GB & & & B
\end{tikzcd}
$$

$$\begin{tikzcd}
X \arrow[r, "\Phi^{-1}(g)"] & GA & & FX \arrow[r, "g"] & A \\
 & {} & & {} \arrow[ll, "\boldsymbol{\Phi^{-1}}"] & \\
Y \arrow[uu, "h"] \arrow[uur, "\Phi^{-1}(g) \circ h"] & & & FY \arrow[uu, "Fh"] \arrow[uur, "g \circ Fh"] &
\end{tikzcd}$$

4.1. **Returning to the Topology on $M(A)$.** Before making progress on adjoints, we add nuance to our discussion of the topology on $M(A)$.

Definition 4.2. A **compact** topological space satisfies the property that every open cover of the topological space has a finite subcover.

Proposition 4.1. *Consider a continuous function f with domain X, where X is compact. $f(X)$, the image of X under f, is compact.*

Proof. Suppose $f : X \to Y$, where X is compact. Take $\bigcup_{i \in I} U_i$, an open cover of $f(X) \in Y$, where U_i is open for all $i \in I$. Then $\bigcup_{i \in I} f^{-1}(U_i)$ must be an open cover of X. We take a finite subcover $J \subset I$, since X is compact. In that case $\bigcup_{i \in J} f^{-1}(U_i)$ covers X, so $\bigcup_{i \in J} f(f^{-1}(U_i))$ covers $f(X)$. Therefore since J is finite, we have obtained a finite subcover of $f(X)$ for any arbitrary open cover and $f(X)$ is compact. $\square$

We will invoke the Bolzano-Weierstrass theorem to show an important corollary.

Fact (Bolzano-Weierstrass). Any compact subset of $\mathbb{R}^n$ is closed and bounded. (See Rosenlicht [6], section III.6.)

Corollary 4.2. *Let X be compact. Then $C(X) = C_b(X)$.*

Proof. Consider $f(X)$, the image of any function $f \in C(X)$. We know that $f(X)$ is compact. But $f : X \to \mathbb{C}$, so $f(X)$ is closed and bounded by the Bolzano-Weierstrauss theorem. As a result, $C(X) \subset C_b(X)$. But $C_b(X) \subset C(X)$ because every bounded continuous function is a continuous function and so lies in $C(X)$. $\square$

Definition 4.3. A **Hausdorff space** is a topological space in which, for every two distinct points x and y, there exist open subsets U, V with $x \in U$ and $y \in V$ such that $U \cap V = \emptyset$, i.e., U and V are disjoint.

Remark. Observe that any metric space X must be Hausdorff. For any two points $x, y \in X$, $x \neq y$, let $d(x, y) = r$. Then the open ball of radius $\frac{r}{2}$ about x will not contain y, and the open ball of radius $\frac{r}{2}$ about y will not contain x; these open balls are open sets. In particular, $\mathbb{C}$ is Hausdorff.

Proposition 4.3. *$M(A)$ is a Hausdorff space.*

Proof. $M(A)$ is Hausdorff from the weak-$*$ topology. Consider any $\varphi, \psi \in M(A)$. We seek to show that we can construct disjoint open sets about φ and ψ respectively. Consider $a \in A$ such that $\varphi(a) \neq \psi(a)$, which exists else the two functions are identical. As $\mathbb{C}$ is a metric space, it is Hausdorff. So we can choose disjoint subsets U_1 and U_2 in $\mathbb{C}$ with $\varphi(a) \in U_1$ and $\psi(a) \in U_2$. In our notation,

$$V_{a,U_j} = \{\lambda \in A^* : \ \lambda(a) \in U_j\}, \text{ with } j = 1, 2.$$

But $V_{a,U_1} \cap V_{a,U_2} = \emptyset$ because if there exists $\lambda \in A^*$ such that $\lambda(a) \in U_1$ and $\lambda(a) \in U_2$, then $U_1 \cap U_2 \neq \emptyset$.

We have, therefore, shown that A^* is Hausdorff. Now, $M(A) \subset A^*$, so $M(A)$ is also Hasudorff because any subset of a Hausdorff space is Hausdorff. □

We need the following lemma.

Lemma 4.4. *Multiplicative linear functionals φ on a unital abelian Banach algebra A to $\mathbb{C}$ have norm 1.*

Proof. Proceed by contradiction. First suppose there exists some multiplicative linear functional φ on A and some $a \in A$ where $\|a\| < 1$, such that $\varphi(a) = 1$. Consider the series $b = \sum_{n\geq 1} a^n$; recall that this series converges whenever $\sum_{n\geq 1} \|a^n\|$ converges, which it will for all a such that $\|a\| < 1$, because in a Banach algebra, $\|a^n\| \leq \|a\|^n$. Then, recall that $a + ab = b$, by dint of how b is constructed; because φ is a multiplicative linear functional, we must have that

$$\varphi(b) = \varphi(a) + \varphi(a)\varphi(b) = \varphi(a) \cdot (1 + \varphi(b)).$$

If $\varphi(a) = 1$, then $\varphi(b) = \varphi(b) + 1$ which is impossible. Now, suppose there exists some φ and a such that $\|a\| \leq 1$ but $|\varphi(a)| > 1$. Let $\varphi(a) = \alpha$; $\varphi(\frac{a}{\alpha}) = 1$. But $\|\frac{a}{\alpha}\| < 1$, and we have shown that this result is impossible. Now, consider $1_A \in A$; $\|1_A\| = 1$ and $\varphi(1_A) = 1 \in \mathbb{C}$ so $|\varphi(1)| = 1$. So we know that there exists $a \in A$ with $\|a\| \leq 1$ such that $\varphi(a) = 1$. Hence

$$\|\varphi\| = \sup\{|\varphi(a)| \text{ for } \|a\| \leq 1\} = 1.$$

□

Proposition 4.5. *$M(A)$ is a closed in the dual space A^* under the weak-$*$ topology.*

Proof. Consider $\varphi \in A^*$ but $\varphi \notin M(A)$. There are two cases. Either φ is not multiplicative or φ is not unital. In either case, we show that we can construct an open set $B \subset A^*$ containing φ such that $B \cap M(A) = \emptyset$, i.e., the complement of $M(A)$ is open.

First suppose φ is not multiplicative. In that case, there exists $a, b \in \mathbb{C}$ such that $\varphi(a)\varphi(b) \neq \varphi(ab)$.

Take U_1, U_2, U_3 as open balls in $\mathbb{C}$ around points $\varphi(a), \varphi(b), \varphi(ab)$ that are sufficiently small that for all $x \in U_1, y \in U_2$, $xy \notin U_3$. It is easy to verify that these open balls exist as long as $\varphi(ab) \neq \varphi(a)\varphi(b)$. Then, consider

$$B = V_{a,U_1} \cap V_{b,U_2} \cap V_{ab,U_3},\ B \subset A^*.$$

Note that $\varphi \in B$ by construction and that B is open. Consider any $\psi \in B$. We show that $\psi \notin M(A)$. Observe $\psi(a) \in U_1$, $\psi(b) \in U_2$, so $\psi(a)\psi(b) \notin U_3$. As $\psi(ab) \in U_3$ by design, then $\psi(a)\psi(b) \neq \psi(ab)$, so ψ is not multiplicative, and therefore $B \cap M(A) = \emptyset$.

Second, let φ be multiplicative but not unital. Then $\varphi(1)\varphi(1) = \varphi(1)$, because φ is multiplicative. If φ is not unital, then $\varphi(1) = 0$. Let $U \subset \mathbb{C}$ be an open ball of radius $\frac{1}{2}$ about 0. Then let $B = V_{1,U}$. Clearly $\varphi \in B$. In that case, let $\psi \in B$. But $\psi(1) \in B$ implies that $\psi(1) \neq 0$ so ψ is not unital. Hence for every non-unital φ, there exists an open set $B \subset A^*$ such that $B \cap M(A) = \emptyset$.

Therefore, $M(A)$ is closed. $\square$

Theorem 4.6. *M(A) is a compact Hausdorff space under the weak-* topology.*

Proof. We invoke the Banach-Alaoglu theorem:

Fact (Banach-Alaoglu). The closed unit ball of the dual space of a normed vector space is compact in the weak-$*$ topology. (See Rudin [7], Theorem 3.15.)

As $\|\varphi\| \leq 1$ for all $\varphi \in M(A)$, $M(A)$ is a closed subset of the unit ball in A^*, which is compact by Banach-Alaoglu. A closed subset of a compact space is compact, so $M(A)$ is compact. $\square$

Remark. An extension of this argument gives that $M(A)$ is locally compact if A is not unital.

4.2. M and C_b as Adjoint Functors.

Theorem 4.7. *The functors C_b and M are **adjoints**. Let X be a topology, and let A be a Banach algebra. Put $f \in Hom_{\mathcal{B}an}(A, C_b(X))$ and $g \in Hom_{\mathcal{T}op}(X, M(A))$. There exists a natural bijection of sets*

$$\Phi : Hom_{\mathcal{B}an}(A, C_b(X)) \to Hom_{\mathcal{T}op}(X, M(A))$$

where $\Phi(f) = g$ and $\Phi^{-1}(g) = f$ such that

$$(f(a))(x) = (g(x))(a)$$

for every $x \in X$, $a \in A$, $f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_b(X))$ and $g \in Hom_{\mathcal{T}\mathbf{op}}(X, MA)$. In particular, the naturality of Φ is equivalent to

(1) $\Phi(C_b k \circ f) = \Phi(f) \circ k$.
(2) $\Phi^{-1}(Mh \circ g) = \Phi^{-1}(g) \circ h$.

Let $f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_bX)$, $g \in Hom_{\mathcal{T}\mathbf{op}}(X, MA)$, $h \in Hom_{\mathcal{B}\mathbf{an}}(B, A)$ and $k \in Hom_{\mathcal{T}\mathbf{op}}(Y, X)$. Recall that from our functors, this means $Mh \in Hom_{\mathcal{T}\mathbf{op}}(MA, MB)$ while $C_bk \in Hom_{\mathcal{B}\mathbf{an}}(C_bX, C_bY)$. If the theorem holds, we should have the following diagrams:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[ddr, "C_bk \circ f"] & C_bX \arrow[dd, "C_bk"] & & MA & X \arrow[l, "\Phi(f)"'] \\
& {} \arrow[rr, "\mathbf{\Phi}"] & & {} & \\
& C_bY & & & Y \arrow[uu, "k"'] \arrow[uul, "\Phi(f) \circ k"]
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\Phi^{-1}(g)"] & C_bX & & MA \arrow[dd, "Mh"] & X \arrow[l, "g"] \arrow[ddl, "Mh \circ g"] \\
& {} & & {} \arrow[ll, "\mathbf{\Phi^{-1}}"'] & \\
B \arrow[uu, "h"] \arrow[uur, "\Phi^{-1}(g) \circ h"'] & & & MB &
\end{tikzcd}$$

In order to prove this theorem, we need to check the following:

(1) $g(x)$ is in $M(A)$ for every $x \in X$.
(2) $g \in Hom_{\mathcal{T}\mathbf{op}}$ if $f \in Hom_{\mathcal{B}\mathbf{an}}$.
(3) $f(a) \in C_b(X)$ for every a in A.
(4) $f \in Hom_{\mathcal{B}\mathbf{an}}$ if $g \in Hom_{\mathcal{T}\mathbf{op}}$.
(5) Φ is a bijection.
(6) Φ is natural.

Proposition 4.8. *If $f : A \to C_b(X)$ is a Banach algebra homomorphism in $Hom_{\mathcal{B}\mathbf{an}}$, $\Phi(f)(x) = g(x) : A \to \mathbb{C}$ is a bounded unital multiplicative linear functional for every*

$x \in X$. In other words, if $f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_b(X))$, then $\Phi(f)(x) \in M(A)$ for every $x \in X$.

Proof. First, observe that $g(x)$ can be described as the evaluation map at x, composed with f, as follows:

$$\begin{aligned} &g(x) : A \to \mathbb{C} \\ &(g(x))(a) = (f(a))(x) = (ev_x \circ f)(a), \end{aligned}$$

where we take ev_x to mean

$$\begin{aligned} &ev_x : C_b(X) \to \mathbb{C} \\ &ev_x(h) = h(x) \end{aligned}$$

for $x \in X$ and $h \in C_bX$. Put otherwise,

$$(\Phi(f))(x) = ev_x \circ f.$$

We claim that $g(x) : A \to \mathbb{C}$ is a unital multiplicative linear functional. Note that

$$\begin{aligned} g(x)(a + \alpha \cdot b) &= (ev_x \circ f)(a + \alpha \cdot b) \\ &= ev_x(f(a + \alpha \cdot b)) \\ &= ev_x(f(a) + f(\alpha \cdot b)) \\ &= (f(a))(x) + (f(\alpha \cdot b))(x) \\ &= (f(a))(x) + \alpha \cdot (f(b))(x) \\ &= g(x)(a) + \alpha \cdot g(x)(b). \end{aligned}$$

A parallel argument holds for the multiplicativity of $g(x)$. We claim that $f(1) = 1_{C_bX} \in C_bX$ for all $f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_bX)$. 1_{C_bX} is defined as the bounded continuous map $h : X \to \mathbb{C}$, $h(x) = 1$ for all $x \in X$. As a result, we see that for all $x \in X$ and $f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_bX)$,

$$(g(x))(1_A) = ev_x(f(1_A)) = 1_{C_bX}(x) = 1 \in \mathbb{C}.$$

To show boundedness, we claim that the evaluation map $ev_x : C_bX \to \mathbb{C}$ is continuous. Fix x_0 and fix ε. Observe that

$$\|h - f\|_\infty = \sup\{|h(x) - f(x)|, \text{ for all } x \in X\} < \varepsilon \implies |h(x_0) - f(x_0)| < \varepsilon$$

so for any $x_0 \in X$, $ev_{x_0} : C_bX \to \mathbb{C}$ is continuous at any $h \in C_bX$.

We are already given

$$f \in Hom_{\mathcal{B}an}(A, C_b(X)),$$

so f, we know, is bounded under the operator norm and therefore continuous. Hence $g(x) = (ev_x \circ f)$ must be continuous as the composition of two continuous functions. Therefore, we have, equivalently, that $g(x)$ is bounded as a linear operator $A \to \mathbb{C}$. Hence we have shown that $g(x) \in MA$. ∎

Proposition 4.9. *If $f : A \to C_b(X)$ is a Banach algebra homomorphism, then*

$$g : X \to M(A)$$

is a continuous function.

Put otherwise, if $f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_b(X))$ then $\Phi(f) \in Hom_{\mathcal{T}\mathbf{op}}(X, M(A))$.

Proof. We rely on the open sets definition of continuity. First, consider the sets U in X which can be described as follows: $U_{f,a,J} = \{x \in X : (f(a))(x) \in J\}$ for $f : A \to C_bX$, $a \in A$, and J, an open set in the usual topology on $\mathbb{C}$. Because $f(a) \in C_b(X)$, $f(a)$ is a continuous map on X, so $U_{f,a,J} = (f(a))^{-1}(J)$, and hence $U_{f,a,J}$ is open in X because J is open in $\mathbb{C}$.

Now consider open sets in MA of the form $V_{a,P} = \{\varphi \in MA : \varphi(a) \in P\}$ by the weak-* topology, with P open in $\mathbb{C}$. Observe that these sets form a topological basis of MA. But

$$\begin{aligned} g^{-1}(V_{a,P}) &= \{x \in X : g(x) \in V_{a,P}\} \\ &= \{x \in X : g(x)(a) \in P\} \\ &= \{x \in X : f(a)(x) \in P\} = U_{f,a,P}, \end{aligned}$$

which is open if P is open. Hence the preimage of any open set of the form $V_{a,P}$ is open in $\mathbb{C}$. So we see that g is continuous, and $g \in Hom_{\mathcal{T}\mathbf{op}}$. □

Proposition 4.10. *Let $g : X \to MA$ be a continuous map between the two topological spaces. Then $(\Phi^{-1}(g))(a) = f(a) : X \to \mathbb{C}$ is a bounded, continuous map, i.e. $f(a) \in C_b(X)$.*

Proof. Fix $a \in A$ and fix g. Define

$$\begin{aligned} &ev_a : MA \to \mathbb{C} \\ &ev_a(\varphi) = \varphi(a). \end{aligned}$$

Then

$$\begin{aligned} &f(a) : X \to \mathbb{C} \\ &(f(a))(x) = (ev_a \circ g)(x). \end{aligned}$$

We claim that ev_a is a continuous map. Recall that φ is a bounded operator of norm 1. So $|\varphi(a)| \leq \|a\|_A$. For any φ, $|ev_a(\varphi)| \leq \|a\|_A \cdot 1$, hence for every $a \in A$, there exists a $C_a \in \mathbb{R}$ such that $|ev_a(\varphi)| \leq C_a\|\varphi\| = C_a$, where $C_a = \|a\|_A$. So $ev_a : MA \to \mathbb{C}$ is bounded under the operator norm and is therefore continuous at φ for all a. Note also that $g : X \to MA$ is continuous. We argue, then, that $f(a) = ev_a \circ g$ is a continuous map, because the composition of continuous functions is continuous.

Note that we have shown $ev_a : MA \to \mathbb{C}$ is a continuous function. Recall that MA is compact, and the images of continuous functions on compact topological spaces are compact. The image of $ev_a : MA \to \mathbb{C}$ must be bounded by some $K \in \mathbb{R}$, since compact subsets of $\mathbb{C}$ are bounded by Bolzano-Weierstrass. In other words, $|ev_a(\varphi)| = |\varphi(a)| < K$ for all $\varphi \in MA$. Now, if we restrict our analysis to only the φ such that $\varphi = g(x)$ for some $x \in X$, we observe that $(ev_a(g(x)) < K$ for all $x \in X$. As a result, we know that $ev_a \circ g$ is a bounded map, or equivalently, that $f(a) = ev_a \circ g$ is a bounded map. Therefore $f(a)$ is a bounded, continuous function. □

Proposition 4.11. *Let $g : X \to MA$ be a continuous map . Then $\Phi^{-1}(g) = f : A \to C_b(X)$ is a unital Banach algebra homomorphism.*

Put otherwise, if $g \in Hom_{\mathcal{T}\mathbf{op}}(X, MA)$ then $\Phi^{-1}(g) = f \in Hom_{\mathcal{B}\mathbf{an}}(A, C_b(X))$.

Proof. We first show that f is a unital algebra homomorphism. Consider $k \in \mathbb{C}$, $a, b \in A$ and $x \in X$. Then

$$f(ka)(x) = (g(x))(ka) = k \cdot (g(x))(a) = k \cdot (f(a))(x).$$

The other algebraic operations are just as simple.

$$f(a \cdot b)(x) = (g(x))(a \cdot b) = (g(x))(a) \cdot (g(x))(b) = (f(a))(x) \cdot (f(b))(x),$$

since $g(x)$ is an algebra homomorphism. The exact same argument works for addition:

$$(f(a + b))(x) = g(x)(a + b) = g(x)(a) + g(x)(b) = (f(a))(x) + (f(b))(x).$$

$f(1_A)(x) = (ev_{1_A} \circ g)(x) = ev_{1_A}(g(x)) = (g(x))(1_A) = 1 \in \mathbb{C}$ as long as $g(x) \in MA$ for all $x \in X$. As a result, as $(f(1_A))(x) = 1 \in \mathbb{C}$ for every x and $f(1_A) \in C_bX$, then $f(1_A) = 1_{C_bX}$.

In summary, $f : A \to C_bX$ is a unital algebra homomorphism.

Now, f is bounded. Every multiplicative linear functional φ in $M(A)$ has $\|\varphi\| = 1$. As a result, we have that each $\|g(x)\| = 1$, since $g(x) \in M(A)$. Suppose we consider $\|a\|_A \leq 1$. Then,

$$|g(x)(a)| \leq \|g(x)\| \|a\|_A \leq 1,$$

but we also know that

$$|(f(a))(x)| = |(g(x))(a)| \leq 1, \text{ for all } x \in X, \text{ for all } a \in A \text{ such that} \|a\|_A \leq 1.$$

Hence, we conclude that $\|f(a)\|_\infty \leq 1$ for $\|a\|_A \leq 1$. Then $\|f\| \leq 1$, so f is bounded with respect to the operator norm. Thus f is a bounded unital algebra homomorphism; i.e., $f \in Hom_{\mathcal{B}\mathbf{an}}$.

□

Proposition 4.12. Φ *is bijective.*

Proof. Let $f : A \to C_bX$ and $z : A \to C_bX$. Let $((\Phi(f))(x))(a) = ((\Phi(z))(x))(a)$ for all $x \in X, a \in A$. In that case,

$$\begin{aligned}
&(ev_x \circ f)(a) = (ev_x \circ z)(a), \text{ for all } x \in X, a \in A. \\
\iff\ & ev_x(f(a)) = ev_x(z(a)), \text{ for all } x \in X, a \in A. \\
\iff\ & (f(a))(x) = (z(a))(x), \text{ for all } x \in X, a \in A. \\
\iff\ & f(a) = z(a), \text{ for all } a \in A. \\
\iff\ & f = z.
\end{aligned}$$

So Φ is injective, because $\Phi(f) = \Phi(z) \implies f = z$. The proof that Φ is surjective is similar. Let $g : X \to MA$ and $p : X \to MA$. Let $(\Phi^{-1}(g))(a)(x) = (\Phi^{-1}(p))(a)(x)$ for all $x \in X$ and $a \in A$. Then,

$$\begin{aligned}
&(ev_a \circ g)(x) = (ev_a \circ p)(x), \text{ for all } x \in X, a \in A. \\
\iff\ & (ev_a(g(x))) = (ev_a(p(x))), \text{ for all } x \in X, a \in A. \\
\iff\ & (g(x))(a) = (p(x))(a), \text{ for all } x \in X, a \in A. \\
\iff\ & g(x) = p(x), \text{ for all } x \in X. \\
\iff\ & g = p.
\end{aligned}$$

Therefore Φ^{-1} is also injective, which means that Φ is bijective. □

Proposition 4.13. Φ *is natural.*

Proof. First, we show $\Phi(C_bk \circ f) = \Phi(f) \circ k$. Recall that $k : Y \to X$, $C_bk : C_bX \to C_bY$ and $f : A \to C_bX$. Now, for all $a \in A$ and $y \in Y$,

$$(\Phi(C_bk \circ f)(y))(a) = ((C_bk \circ f)(a))(y) = (C_bk(f(a)))(y) = (f(a))(k(y)),$$

by how C_bk is defined. Note that $f(a) \in C_bX$, and $k(y) \in X$, so $f(a)$ can indeed accept the element $k(y)$. We proceed and observe that

$$f(a)(k(y)) = ((\Phi(f))(k(y)))(a) = (\Phi(f))(k(y))(a) = ((\Phi(f) \circ k)(y))(a),$$

where $\Phi(f) \circ k : Y \to MA$. In particular, because $\Phi(f) : X \to MA$, we observe that $\Phi(f) \circ k : Y \to MA$, as $k : Y \to X$. As a result, for all $y \in Y, a \in A$,

$$\Phi(C_bk \circ f) = \Phi(f) \circ k.$$

Next, we show $\Phi^{-1}(Mh \circ g) = \Phi^{-1}(g) \circ h$. Let $h : B \to A$, so $Mh : MA \to MB$, while $g : X \to MA$. In that case, for all $b \in B$ and $x \in X$,

$$(\Phi^{-1}(Mh \circ g))(b)(x) = ((Mh \circ g)(x))(b) = (Mh(g(x)))(b) = (g(x))(h(b)).$$

Then

$$g(x)(h(b)) = (((\Phi^{-1}(g))(h(b)))(x)) = (\Phi^{-1}(g))(h(b))(x) = ((\Phi^{-1}(g) \circ h)(b))(x)$$

for all $b \in B$ and $x \in X$, where $\Phi^{-1}(g) \circ h : B \to C_b X$. In particular, because $\Phi^{-1}(g) : A \to C_b X$, and $h : B \to A$, we note that $\Phi^{-1}(g) \circ h$ accepts elements in B and returns elements in $C_b X$. Therefore, we have shown that $\Phi^{-1}(Mh \circ g) = \Phi^{-1}(g) \circ h$. $\square$

As a result, we have verified each argument necessary to conclude that Theorem 4.7 holds.

5. Results from our Adjoints

5.1. **The Gelfand Transform.** We have shown that the functors C_b and M are adjoints. In this section, we discuss the natural transformations that are given by the functor

$$C_b M : \mathcal{B}\mathbf{an} \to \mathcal{B}\mathbf{an}.$$

Note that we do not invoke any specific properties of C_b or M in this chapter, but merely rely on the properties of the natural bijection Φ.

Put $X = MA$. Then we have a natural bijection

$$\Phi : Hom_{\mathcal{B}\mathbf{an}}(A, C_b MA) \to Hom_{\mathcal{T}\mathbf{op}}(MA, MA).$$

Let I_{MA} denote the identity map

$$\begin{aligned} &I_{MA} : MA \to MA \\ &I_{MA}(\varphi) = \varphi. \end{aligned}$$

The adjunction Φ gives us a corresponding Banach algebra homormorphism $\Gamma_A = \Phi^{-1}(I_{MA})$. By definition,

$$(\Gamma_A(a))(\varphi) = I_{MA}(\varphi)(a) = \varphi(a)$$

for $a \in A,\ \varphi \in MA$. Put simply, $\Gamma_A = \Phi^{-1}(I_{MA})$ is the map

$$\begin{aligned} &\Gamma_A : A \to C_b(M(A)) \\ &\Gamma_A(a) = ev_a \end{aligned}$$

where ev_a is the evaluation map

$$\begin{aligned} &ev_a : M(A) \to \mathbb{C} \\ &ev_a(\varphi) = \varphi(a). \end{aligned}$$

The Banach algebra homomorphism Γ_A is called the Gelfand transform.

Proposition 5.1. *The Gelfand transform is natural in the following precise sense. Suppose A, B are unital abelian Banach algebras. Let $p \in Hom_{\mathcal{B}\mathbf{an}}(A, B)$, with $Mp \in Hom_{\mathcal{T}\mathbf{op}}(MB, MA)$ and $C_b Mp \in Hom_{\mathcal{B}\mathbf{an}}(C_b MA, C_b MB)$. Let $\Gamma_A : A \to C_b MA$ and $\Gamma_B : B \to C_b MB$ be the Gelfand transforms for A and B respectively. Then $\Gamma_B \circ p = C_b Mp \circ \Gamma_A$. In other words, we can draw up the following diagram:*

$$\begin{tikzcd}
A \arrow[r, "\Gamma_A : a \mapsto ev_a"] \arrow[d, "p"] & C_bMA \arrow[d, "MC_bp"] \\
B \arrow[r, "\Gamma_B : b \mapsto ev_b"] & C_bMB
\end{tikzcd}$$

Proof. Employ the fact that Φ is natural and that $\Gamma_A = \Phi^{-1}(I_{MA})$. Substitute C_bMA for C_bX, Γ_A for f and C_bMp for C_bk in the original adjoint diagrams to obtain these diagrams:

$$\begin{tikzcd}
A \arrow[r, "\Gamma_A"] \arrow[dr, "C_bMp \circ \Gamma_A"'] & C_bMA \arrow[d, "C_bMp"] \\
 & C_bMB
\end{tikzcd}
\quad \xrightarrow{\Phi} \quad
\begin{tikzcd}
MA & MA \arrow[l, "I"'] \\
 & MB \arrow[ul, "Mp"] \arrow[u, "Mp"']
\end{tikzcd}$$

$$\begin{tikzcd}
B \arrow[r, "\Gamma_B"] & C_bMB \\
A \arrow[u, "p"] \arrow[ur, "\Gamma_B \circ p"] &
\end{tikzcd}
\quad \xrightarrow{\Phi} \quad
\begin{tikzcd}
MB \arrow[d, "Mp"] & MB \arrow[l, "I"'] \arrow[dl, "Mp"] \\
MA &
\end{tikzcd}$$

We can produce these diagrams by applying the natural map Φ. That is, we know by the naturality of the adjoint that

$$\Phi(C_bMp \circ \Gamma_A) = \Phi(\Gamma_A) \circ Mp = I_{MA} \circ Mp = Mp$$

simply because $\Phi(\Gamma_A) = I_{MA}$ by construction.

Similarly, $\Phi(\Gamma_B \circ p) = Mp \circ \Phi(\Gamma_B) = Mp \circ I_{MB} = Mp$.

Now, observe that Φ is a bijection, so it is injective. As $\Phi(\Gamma_B \circ p) = \Phi(C_b Mp \circ \Gamma_A)$, we conclude that $\Gamma_B \circ p = C_b Mp \circ \Gamma_A$. $\square$

5.2. **The Stone-Cech Transform.** We can obtain the analog of the Gelfand transform by considering the composition of functors

$$MC_b : \mathcal{T}\mathbf{op} \to \mathcal{T}\mathbf{op}.$$

By using Φ, we notice a transform from a topological space to a compact Hausdorff space in the following way. Put $A = C_b X$. Then there is a natural bijection

$$\Phi : Hom_{\mathcal{B}\mathbf{an}}(C_b X, C_b X) \to Hom_{\mathcal{T}\mathbf{op}}(X, MC_b X).$$

Let $I_{C_b X}$ be the identity map in $Hom_{\mathcal{B}\mathbf{an}}(C_b X, C_b X)$. From our adjunction, we can identify a corresponding continuous map $\Psi_X \in Hom_{\mathcal{T}\mathbf{op}}(X, MC_b X)$, $\Psi_X = \Phi(I_{C_b X})$ defined by

$$\Psi_X(x)(h) = I_{C_b X}(h)(x) = h(x) \in \mathbb{C}.$$

Then Ψ_X is the continuous map which satisfies

$$\Psi_X : X \to MC_b X$$
$$\Psi_X(x) = ev_x$$

where

$$ev_x : C_b X \to \mathbb{C}$$
$$ev_x(h) = h(x)$$

We call Ψ_X the Stone-Cech transform on X.

Proposition 5.2. *The Stone-Cech transform is natural in the following precise sense. Let X and Y be topological spaces. Let $f \in Hom_{\mathcal{T}\mathbf{op}}(X, Y)$, with $C_b f \in Hom_{\mathcal{B}\mathbf{an}}(C_b Y, C_b X)$ and $MC_b f \in Hom_{\mathcal{T}\mathbf{op}}(MC_b X, MC_b Y)$. Let $\Psi_X : X \to MC_b X$ and $\Psi_Y : Y \to MC_b Y$ be Stone-Cech transforms on X and Y. Then $\Psi_Y \circ f = MC_b f \circ \Psi_X$. In other words, we obtain the following diagram:*

$$\begin{tikzcd}
X \arrow[r, "\Psi_X : x \mapsto ev_x"] \arrow[d, "f"] & MC_b X \arrow[d, "MC_b f"] \\
Y \arrow[r, "\Psi_Y : y \mapsto ev_y"] & MC_b Y
\end{tikzcd}$$

Proof. The argument is symmetric to the proof in the prior section. □

5.3. **A Bijection on $M(A)$.** For A, a unital abelian Banach algebra, let $L(A)$ represent the set of the maximal ideals of A. In this section, we need the fact that $L(A) \cong M(A)$; that is, there exists a bijection between the set of maximal ideals on A and the set of bounded unital multiplicative linear functionals on A.

Theorem 5.3. *Consider $L(A)$, the set of maximal ideals of a unital abelian Banach algebra A. Consider $M(A) = Hom_{\mathcal{B}\mathbf{an}}(A, \mathbb{C})$. $L(A) \cong M(A)$; that is, the map*

$$M(A) \to L(A)$$
$$\varphi \mapsto ker(\varphi)$$

is a bijection.

Proof. Elementary facts about ideals can be found in Herstein [3], sections 4.3 and 4.4. Take $\varphi \in M(A)$. First we see that a non-trivial $\varphi : A \to \mathbb{C}$ must be surjective. Recall from the First Homomorphism Theorem that if we put $K = ker(\varphi)$, the surjective map $\varphi : A \to \mathbb{C}$ determines an isomorphism $A/K \cong \mathbb{C}$. By another algebraic fact, since $\mathbb{C}$ is a field, it follows that $ker(\varphi)$ is a maximal ideal of A. Thus the map $M(A) \to L(A)$ is well-defined.

The map $M(A) \to L(A)$ is injective. Suppose $ker(\varphi) = ker(\psi)$. We argue that $\varphi = \psi$.

Invoking the First Homomorphism Theorem, if we are given $K = ker(\varphi)$ then $\varphi : A \to \mathbb{C}$ is the composition of the quotient map $A \to A/K$ and the choice of an isomorphism $A/K \cong \mathbb{C}$. So $\varphi \neq \psi$ if and only if this second isomorphism differs for φ and ψ. But there can not be more than one isomorphism $A/K \cong \mathbb{C}$, because otherwise there would be a nontrivial *automorphism* of $\mathbb{C} \cong \mathbb{C}$. But since we are dealing with unital algebra homomorphisms, this automorphism would have to be linear in $\mathbb{C}$. There is clearly only one automorphism that satisfies $\mathbb{C}$-linearity, as $\psi(1_{A/K}) = \varphi(1_{A/K}) = 1_{\mathbb{C}}$, which determines every other value for ψ and φ. As a result, $\varphi = \psi$.

The map $M(A) \to L(A)$ is surjective. We need to use a number of facts from the theory of Banach algebras.

Fact. A maximal ideal in a Banach algebra is necessarily closed. (See Rudin [7], Proposition 11.2.)

Fact. Let N be a closed ideal in A. Consider $a \in A$ and the equivalence class $[a] = a + N = \{a + x : x \in N\}$. A norm on elements $[a] \in A/N$ is defined as follows:

$$\|[a]\|_{A/N} = \inf_{x \in N} \|a + x\|_A.$$

Then A/N is a Banach algebra under $\|\cdot\|_{A/N}$. (See Rudin [7], Theorem 1.41.)

Fact. The only simple abelian Banach algebra is $\mathbb{C}$. (See Davidson [1], Theorem I.2.4.)

Now choose a maximal ideal N in A. N is an element in the set $L(A)$. By the first and second fact, N is closed and the quotient A/N is a unital abelian Banach algebra. Since N is maximal, A/N is a field, and therefore simple. Therefore by the third fact, $A/N \cong \mathbb{C}$. Thus N is the kernel of the quotient map $A \to A/N \cong \mathbb{C}$, which is a unital Banach algebra homormorphism and so in $M(A)$. This proves that our map $M(A) \to L(A)$ is surjective: for every $N \in L(A)$, there exists $\varphi \in M(A)$ such that $N = ker(\varphi)$. □

5.4. **The Stone-Cech Transform Ψ_X as a bijection.**

Theorem 5.4. *If X is compact Hausdorff, then the Stone-Cech transform*

$$\Psi_X : X \to MC_bX$$

is a bijection.

Proof. We claim first that

Lemma 5.5. *Let X be a compact topological space. Then if $I \subset C(X)$ is a proper ideal, there exists some $x \in X$ such that $f(x) = 0$ for all $f \in I$.*

Suppose to the contrary that the lemma is false. Then, for any $x \in X$, we can choose $f_x \in I$ such that $f_x(x) \neq 0$. Consider the subsets $U_x \subset \mathbb{C}$, $U_x = f_x^{-1}(\mathbb{C}\backslash\{0\})$. Each U_x consists, then, of the elements in X where the function f_x is non-zero. We claim that $\{U_x : x \in X\}$ is an open cover of X, since for every $x \in X$, $x \in U_x$.

Now, X is compact, so we can take a finite subcover $\{U_1, U_2, \dots, U_n\}$, which yields a corresponding set of functions $\{f_1, f_2, f_3, \dots, f_n\}$. For each $x \in X$, there exists $i \leq n$ such that $f_i(x) \neq 0$, because x is contained in at least one set U_i.

Each f_i is in our ideal I. We know, then, that $\bar{f}_i \cdot f_i = |f_i|^2$ is in I, by the nature of ideals. Since ideals are also closed under addition, there exists a function g in the ideal where:

$$g = \sum_i^n |f_i|^2 = |f_1|^2 + \cdots + |f_n|^2.$$

Observe that $g(x) > 0$ for all $x \in X$, as g is the sum of squares and one of the terms is nonzero for every x. We have, then, that g has a multiplicative inverse $\frac{1}{g} \in C(X)$. Now, $\frac{1}{g} \cdot g = 1_{C_bX}$, so the unit is in the ideal I. But this means that $f \cdot 1 \in I$ for all $f \in C(X)$. Hence we derive a contradiction: I is not a proper ideal. The lemma is proven.

Proceed to consider I, a maximal ideal in $C(X)$. Now, from the lemma, there exists $x \in X$ such that $f(x) = 0$ for every $f \in I$. In that case, notice that ideals of the form

$$I_x = \{f \in C(X) : f(x) = 0\}$$

are proper ideals in $C(X)$, and that $I \subseteq I_x$, because $f(x) = 0$ for all $f \in I$, so I can only be a restriction of I_x and cannot contain any function that is not in I_x. But as I is maximal, we must have that $I = I_x$; hence, each maximal ideal in $C(X)$ is of the form I_x. We argue that every ideal of the form I_x is maximal. Suppose there exists an ideal J where $I_x \subset J$. From the lemma, we know that there exists a point $y \in X$ such that $f(y) = 0$ for all $f \in J$. In order to generate a contradiction, we suppose that $x \neq y$. In that case, because X is a compact Hausdorff space and therefore normal, from the Tietze extension theorem (see Munkres [5], Theorem 3.2), there exists a continuous $f \in C(X)$ such that $f(y) = 1$ and $f(x) = 0$, which means that $f \in I_x$ but $f \notin J$. In that case, I_x is not contained in J, so we have derived a contradiction. Then $x = y$; that is, I_x is a maximal ideal.

We have shown that there is a bijection Λ where

$$\begin{aligned}
&X \to L(C(X)) \\
&\Lambda(x) = I_x = \{f \in C(X) : f(x) = 0\}.
\end{aligned}$$

By construction, I_x is the kernel of the evaluation map $ev_x : C(X) \to \mathbb{C}$. But from the fact above, the kernel map taking $M(C(X)) \to L(C(X))$ is a bijection. Then, composing bijections, we see that the map

$$\begin{aligned}
&X \to MC_bX \\
&x \mapsto ev_x
\end{aligned}$$

must be a bijection. But this is precisely the Stone-Cech tranform; that is $\Psi_X(x) = ev_x$. As a result, $\Psi_X(x) = ev_x$ must be a bijection. $\square$

Definition 5.1. Let X and Y be topological spaces, and let $f : X \to Y$ be a continuous map. If f has a continuous inverse map, f is called a **homeomorphism**.

Corollary 5.6. *If X is a compact Hausdorff space, then the Stone-Cech transform*

$$\Psi_X : X \to MC_bX$$

is a homeomorphism.

We know that Ψ_X is a continuous bijection between two topological spaces. Employ the topological fact

Fact. If X is compact and Y is Hausdorff, a continuous bijection $f : X \to Y$ is a homeomorphism. (See Munkres [5], Theorem 5.6.)

6. Equivalence of Categories

6.1. **Two Subcategories.** The functors M and C_b do not define an equivalence of the categories $\mathcal{B}$**an** and $\mathcal{T}$**op**. In this section, we isolate a *subcategory* of $\mathcal{B}$**an** and a subcategory of $\mathcal{T}$**op** which are equivalent. These subcategories happen to consist of the objects in the image of the Gelfand and Stone-Cech transforms, as well as the set of morphisms between these objects.

We first consider two new objects.

Definition 6.1. The collection of objects generated by applying the functor composition MC_b to topological spaces, together with continuous functions between topological spaces of this form, is a category, which we shall denote $\mathcal{SC}$.

Observe that MC_bX is a topological space because C_bX is a unital abelian Banach algebra. We can certainly limit the objects in the category $\mathcal{T}$**op** to objects of the form MC_bX, and to continuous functions between objects of this type.

Definition 6.2. The collection of objects generated by applying the functor composition C_bM to abelian Banach algebra abelian Banach algebra A, together with the unital Banach algebra homomorphisms between objects of this form, form a category, which we shall denote $\mathcal{GN}$.

By the exact same argument, $\mathcal{GN}$ is a category. For C_bMA is a unital abelian Banach algebra, as C_b is a functor from $\mathcal{T}$**op** to $\mathcal{B}$**an** and MA is a compact Hausdorff space. By the reasoning above, we can limit a subcategory to those morphisms which land in the category's objects without trouble.

6.2. **Natural Transformations.** We build up to the definition of category equivalence by introducing the notion of a natural transformation.

Definition 6.3. Consider functors F and G and categories C, D, where $F : C \to D$ and $G : C \to D$. A **natural transformation** from F to G is a collection of morphisms η which satisfies:

(1) For each object X in Obj_C, there exists $\eta_X : F(X) \to G(X)$, where η_X is called a "component" of η.
(2) The components must commute such that $\eta_Y \circ F(f) = G(f) \circ \eta_X$, for every morphism $f \in Hom_C(X, Y)$.

In particular, we have that the following picture:

$$\begin{tikzcd}
F(X) \arrow[r, "\eta_X"] \arrow[d, "F(f)"] & G(X) \arrow[d, "G(f)"] \\
F(Y) \arrow[r, "\eta_Y"] & G(Y)
\end{tikzcd}$$

Definition 6.4. Let C be a category, and let X, Y be objects. A morphism f $Hom_C(X, Y)$ is an **isomorphism** if there exists $g \in Hom_C(Y, X)$ such that

$$g \circ f = I_X \text{ and } f \circ g = I_Y.$$

Definition 6.5. We say that the natural transformation is a **natural isomorphism** if each η_X is an isomorphism.

Definition 6.6. A functor $F : C \to D$ yields an **equivalence of categories** if there exists a functor $G : D \to C$ and natural isomorphisms $FG \cong I_D : D \to D$ and $GF \cong I_C : C \to C$, where I denotes the identity functor.

Note that, by reversing arrows, if F is an equivalence of categories, so is G.

6.3. **Idempotency of the Transforms.**

Proposition 6.1. *The functor $MC_b : \mathcal{T}\mathbf{op} \to \mathcal{T}\mathbf{op}$ is idempotent in the sense that $MC_b \cong MC_bMC_b$ are naturally isomorphic functors. Furthermore, $C_bM : \mathcal{B}\mathbf{an} \to \mathcal{B}\mathbf{an}$ is idempotent in the sense that $C_bM \cong C_bMC_bM$ are naturally isomorphic functors.*

Proof. Observe that MC_bX is a compact Hausdorff space for any topological space X, because C_bX is an object in $\mathcal{B}\mathbf{an}$. As a result, $\Psi_{MC_b} : MC_bX \to MC_bMC_bX$ is a homeomorphism, as we have shown in the previous section. Then, from Proposition 5.2, we see that Ψ_{MC_bX} gives the components of a natural isomorphism $\eta : MC_b \cong MC_bMC_b$.

Let A be a unital abelian Banach algebra A, where MA is a compact Hausdorff space. We invoke the fact that functors map isomorphisms to isomorphisms. Then $\Psi_{MA} : MA \to MC_bMA$ is a homeomorphism, so $C_b(\Psi_{MA}) : C_bMC_bMA \to C_bMA$ is an isomorphism of unital Banach algebras. Then, in that case, $\epsilon_A = C_b(\Psi_{MA})$ supplies the components of a natural isomorphism $\epsilon : C_bMC_bM \cong C_bM$. That ϵ is natural is a result of the naturality of Ψ_X, as C_b is a contravariant functor. □

Corollary 6.2. *The Stone-Cech transform is a homeomorphism for objects in $\mathcal{SC}$. The Gelfand transform is a unital Banach algebra isomorphism for objects in $\mathcal{GN}$.*

Proof. The first statement is given in Proposition 6.1. To show the second statement, we argue that ϵ_A is simply the inverse of $\Gamma_{C_bMA} : C_bMA \to C_bMC_bMA$. Let $B = C_bMA$ and put $b \in B$. Then we know that $\epsilon_A(\Gamma_B(b)) = C_b(\Psi_{MA})(\Gamma_B(b))$. We aim to show $\epsilon_A(\Gamma_B(b)) = b$. Observe that $\Gamma_B(b) = ev_b$. But $C_b(\Psi_{MA})(ev_b) = ev_b \circ \Psi_{MA}$, where $ev_b \circ \Psi_{MA} : MA \to \mathbb{C}$. We argue, then, that $b = ev_b \circ \Psi_{MA}$, with $b \in B = C_bMA$. Choose $\varphi \in MA$. $\Psi_{MA}(\varphi) = ev_\varphi : C_bMA \to \mathbb{C}$. Then

$$(ev_b \circ \Psi_{MA})(\varphi) = ev_b(\Psi_{MA}(\varphi)) = ev_b(ev_\varphi) = ev_\varphi(b) = b(\varphi),$$

which proves that $(ev_b \circ \Psi_{MA})(\varphi) = b(\varphi)$. As a result, we have shown that $\epsilon_A(\Gamma_B(b))(\varphi) = b(\varphi)$ for any $\varphi \in MA$ so $\epsilon_A(\Gamma_B(b)) = b$, as desired. ☐

6.4. **Equivalence of $\mathcal{SC}$ and $\mathcal{GN}$.**

Theorem 6.3. *The functors $C_b : \mathcal{SC} \to \mathcal{GN}$ and $M : \mathcal{GN} \to \mathcal{SC}$ are equivalences of categories. More precisely, the Gelfand transform is a natural isomorphism of functors $I_{\mathcal{GN}} \cong C_bM$ and the Stone-Cech transform is a natural isomorphism $I_{\mathcal{SC}} \to MC_b$.*

Proof. For any object Y in $\mathcal{SC}$, $Y = MC_bX$ for some topological space X. As a result, we know from the work in Proposition 6.1 that the Stone-Cech transform $\Psi_Y : Y \to MC_bY$ is a homeomorphism that determines the components of a natural transformation $\Psi : I_{\mathcal{SC}} \cong MC_b$. For any B in $\mathcal{GN}$, $B = C_bMA$ for some unital abelian Banach algebra A. Therefore we conclude that the Gelfand transform $\Gamma_B : C_bMB$ is an isomorphism that determines the components of a natural transformation $\Gamma : I_{\mathcal{GN}} \cong C_bM$. ☐

6.5. **The Category $\mathcal{SC}$ and the Category of Compact Hausdorff Spaces.**

Definition 6.7. The objects which are Hausdorff spaces, together with the morphisms that are continuous functions between compact Hausdorff spaces, form a category, denoted $\mathcal{H}$.

As before, we limit a category to objects in the category and the morphisms between these objects. In this case, we take a subcategory of $\mathcal{T}$**op**.

Theorem 6.4. *The composition of functors $MC_b : \mathcal{H} \to \mathcal{SC}$ yields an equivalence of categories.*

Proof. Let $F : \mathcal{SC} \to \mathcal{H}$ be the inclusion functor; that is, for $Y = MC_bX$ (with X a topological space) $F(Y)$ merely picks out the identical Y in $\mathcal{H}$, while $F(f)$ merely picks out the identical morphism in $\mathcal{H}$. Then for any compact Hausdorff space, the Stone-Cech transform yields the components of the natural isomorphism $I_{\mathcal{H}} \cong FMC_b$ because for any Y in $\mathcal{H}$, Ψ_Y is a homeomorphism. Furthermore, from our idempotency argument, we have determined that the Stone-Cech transform is a natural isomorphism $I_{\mathcal{SC}} \cong MC_bF$. ☐

Corollary 6.5. *The subcategory $\mathcal{GN}$ of $\mathcal{B}$**an** is equivalent to the category of compact Hausdorff spaces.*

Note that category equivalence is transitive.

7. C*-algebras

7.1. **The Category $\mathcal{C}$st.** We introduce the following definitions to build up to the definition of a C*-algebra.

Definition 7.1. A $*$-algebra is a complex algebra A endowed with an adjoint map

$$A \to A \qquad a \mapsto a^*$$

that satisfies the following properties:

Linear in Sums: $(a+b)^* = a^* + b^*$
Conjugate Linear: $(\lambda a)^* = \bar{\lambda} a^*$
Involution: $(a^*)^* = a$
Reverses Products: $(ab)^* = b^* a^*$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

These axioms are modeled on the properties of the adjoint (the complex conjugate transpose) of a complex matrix.

Definition 7.2. A **Banach*-algebra** is a Banach algebra endowed with an adjoint that is norm preserving, i.e. $\|a^*\| = \|a\|$.

Definition 7.3. A Banach*-algebra A is a **C*-algebra** if:
C*-identity: $\|a\|^2 = \|a^* a\|$ for all $a \in A$.

Definition 7.4. A **$*$-homomorphism** $f : A \to B$ between two $*$-algebras is an algebra homomorphism that preserves the adjoint, i.e. $f(a^*) = f(a)^*$.

This gives us an obvious new category.

Definition 7.5. $\mathcal{C}$**st**, the category of C*-algebras, consists of unital commutative C*-algebras (the objects) together with unital $*$-homomorphisms.

7.2. $C_b(X)$ **as a C*-Algebra.** If X is a topological space, then we have already shown that $C_b(X)$ is a unital abelian Banach algebra. We argue that complex conjugation forms an adjoint on $C_b(X)$. That is, for $f \in C_b(X)$,

$$f^*(x) = \overline{f(x)}$$

It is trivial that complex conjugation is conjugate linear, involutive, and norm preserving. The only fact requiring proof is that complex conjugation reverses products. Let $f, g \in C_b(X)$. Then $\overline{(fg)(x)} = \overline{f(x)} \cdot \overline{g(x)}$, if we invoke the basic fact that $\overline{ab} = \overline{a} \cdot \overline{b}$

for all $a, b \in \mathbb{C}$. Therefore $(fg)^* = f^*g^*$. But as $C_b(X)$ is commutative, we know that $f^*g^* = g^*f^*$, and we are done.

Theorem 7.1. *If X is a topological space, $C_b(X)$ with complex conjugation as an adjoint is a C*-algebra.*

Proof. We claim that the adjoint satisfies the C*-identity. Recall that the norm for $C_b(X)$ is the supremum norn $\|\cdot\|_\infty$. Now,

$$\|f^*f\|_\infty = \sup_{x\in X}\{|\overline{f(x)}f(x)|\} = \sup_{x\in X}\{|f(x)|^2\} = \left(\sup_{x\in X}\{|f(x)|\}\right)^2 = \|f\|_\infty^2$$

We merely recall that $\overline{a}a = |a|^2$ for every $a \in \mathbb{C}$, and that the supremum of the squares of a set of nonnegative real numbers equals the square of the supremum. □

In the next lemma, we complete the argument that we may think of C_b as a functor $\mathcal{T}\mathbf{op} \to \mathcal{C}\mathbf{st}$. We have shown that $C_b(X)$ is a C*-algebra if X is in $\mathcal{T}\mathbf{op}$; now, we need to show that $C_b(f)$ is a C*-algebra homomorphism if $f \in Hom_{\mathcal{T}\mathbf{op}}$.

Lemma 7.2. *Let $f : X \to Y$ be a continuous map between topological spaces X, Y. Then $C_b(f) : C_b(Y) \to C_b(X)$ is a $*$-homomorphism.*

Proof. By definition, $C_b(f)(g) = g \circ f$ for $g \in C_b(Y)$. So for every $x \in X$,

$$(C_b(f)(g^*))(x) = (g^* \circ f)(x) = g^*(f(x))) = \overline{g(f(x))} = \overline{(C_b(f)(g))(x)}$$

As a result, $C_b(f)(g^*) = (C_b(f)(g))^*$, and so $C_b(f)$ is a $*$-homomorphism. □

In fact, we require the following stronger result.

Lemma 7.3. *Let X, Y be compact Hausdorff spaces. Every unital Banach algebra homomorphism $f : C(X) \to C(Y)$ is a $*$-homomorphism.*

We first argue that the Gelfand transform $\Gamma_{CX} : CX \to CMCX$ is a $*$-isomorphism. Consider $g \in CX$. Then $\Gamma_{CX}(g) = ev_g : MCX \to \mathbb{C}$. But any element in MCX must be of the form $ev_x : CX \to \mathbb{C}$, because

$$\Psi_X : X \to MC_bX$$
$$x \mapsto ev_x$$

is surjective. In that case, $ev_g(ev_x) = ev_x(g) = g(x)$, so $\Gamma_{CX}(g)(ev_x)^* = g(x)^* = g^*(x)$ and then we can see that Γ_{CX} is a $*$-isomorphism.

Take an arbitrary $f : C(X) \to C(Y)$. Now, recall from the naturality of the Gelfand transform that $CM(f) \circ \Gamma_{CX} = \Gamma_{CY} \circ f$. We have previously shown that $CM(f)$ preserves the adjoint, as does Γ_{CX} and Γ_{CY}. Then for any $z \in CX$, $(CM(f) \circ \Gamma(CX))(z^*) = (CM(f) \circ \Gamma(CX))(z)^*$, so we can take the adjoint on the right-hand side as well to obtain that

$$(\Gamma_{CY} \circ f)(z^*) = (\Gamma_{CY} \circ f)(z)^*.$$

In that case,

$$\Gamma_{CY}(f(z^*)) = \Gamma_{CY}(f(z))^* = \Gamma_{CY}(f(z)^*)$$

as Γ preserves the adjoint. In that case, because Γ_{CY} is an isomorphism, we can conclude that $f(z)^* = f(z^*)$, and we are done.

7.3. **The Category $\mathcal{GN}$ and C*-algebras.** We can think of $\mathcal{C}\mathbf{st}$ as a subcategory of $\mathcal{B}\mathbf{an}$. In particular, if we 'forget' the adjoint, then an object in $\mathcal{C}\mathbf{st}$ is just a unital abelian Banach algebra, i.e. an object in $\mathcal{B}\mathbf{an}$. Every morphism in $\mathcal{C}\mathbf{st}$ is certainly a morphism in $\mathcal{B}\mathbf{an}$, and so forgetting the adjoint is a functor from $\mathcal{C}\mathbf{st}$ to $\mathcal{B}\mathbf{an}$, we can obtain the 'forgetful functor'

$$Z : \mathcal{C}\mathbf{st} \to \mathcal{B}\mathbf{an}.$$

It is clear that the functor preserves identity maps and composition, because it merely removes mathematical structure extraneous to these questions. Now, let us compose the forgetful functor Z with the Gelfand transform C_bM. In that case, we obtain a functor

$$C_bMZ : \mathcal{C}\mathbf{st} \to \mathcal{GN}.$$

The prior section illustrates that there is a functor

$$Y : \mathcal{GN} \to \mathcal{C}\mathbf{st}.$$

In particular, every object in $\mathcal{GN}$ is a $\mathcal{C}\mathbf{st}$ algebra, because every object is of the form C_bMA, and MA is a topological space. We have argued that any morphism $f : C_bMA \to C_bMB$ is a $*$-homomorphism as MA and MB are compact Hausdorff. In that case, the functor Y merely recovers the structure hidden in the background. It is clear that identity will be preserved and so will composition of maps, because $Y(f) = f$ for all $f \in Hom_{\mathcal{GN}}$. Both Z and Y are covariant functors.

Why introduce these functors? They serve as technical details necessary to obtain this paper's final result.

Theorem 7.4. *The Gelfand transform $C_bMZ : \mathcal{C}\mathbf{st} \to \mathcal{GN}$ is an equivalence of categories.*

Proof. We show that $C_bMZ \circ Y \cong I_{\mathcal{GN}}$ and $Y \circ C_bMZ \cong I_{\mathcal{C}\mathbf{st}}$ where I is the identity functor.

First, $C_bMZY \cong I_{\mathcal{GN}}$. This has already been proven. The composition $ZY : \mathcal{GN} \to \mathcal{B}\mathbf{an}$ is just the inclusion of $\mathcal{GN}$ as a subcategory of $\mathcal{B}\mathbf{an}$ (where we first introduce an adjoint, and then forget it again). It does no work. Thus C_bMZY is the same as

$C_bM : \mathcal{GN} \to \mathcal{GN}$. We have already shown that $C_bM \cong I_{\mathcal{GN}}$, which follows from the idempotency of the Gelfand transform C_bM. (See Proposition 6.1 and Corollary 6.3.)

Now, we must show that $YC_bMZ \cong I_{\mathcal{C}\mathbf{st}}$. As the Gelfand transform is natural, and Y and Z are not changing the composition of maps in any meaningful sense, it is clear that there is a natural transform linking these two functors. What is highly non-trivial is that the Gelfand-transform

$$T \to YC_bM(ZT)$$

is an isomorphism in the category $\mathcal{C}\mathbf{st}$.

We employ the Gelfand-Naimark Theorem, which we prove in the next section, to complete the paper.

Fact (Gelfand-Naimark). Let T be a unital commutative C*-algebra. Then $\Gamma_{ZT} : ZT \to C_bM(ZT)$ is an isometric $*$-isomorphism.

With that fact, we will have that $YC_bMZ \cong I_{\mathcal{C}\mathbf{st}}$. For every T in $\mathcal{C}\mathbf{st}$, $Y(\Gamma_{ZT})$ will give the components of a natural isomorphism. The transform is natural because Γ is natural. And if the Gelfand-Naimark theorem holds, then each component is an isomorphism in the category $\mathcal{C}\mathbf{st}$. □

Thanks to the Gelfand-Naimark Theorem, the content of the abstract categories $\mathcal{GN}$ and $\mathcal{SC}$ is now made more explicit.

Corollary 7.5. *The category $\mathcal{C}\mathbf{st}$ of unital commutative C*-algebras is equivalent to the category $\mathcal{H}$ of compact Hausdorff spaces.*

Therefore, the Gelfand-Naimark theorem permits us to identify two very different categories.

8. The Gelfand-Naimark Theorem

In this final section we suppress the functors Y and Z from our notation. Thus, we treat M as a functor from $\mathcal{C}\mathbf{st} \to \mathcal{T}\mathbf{op}$, and we call C_b a functor from $\mathcal{T}\mathbf{op} \to \mathcal{C}\mathbf{st}$. Note that the arguments in the prior chapter permit this switch in notation.

Our objective in this chapter is to prove the Gelfand-Naimark theorem.

Theorem 8.1 (Gelfand-Naimark)**.** *Let A be a unital commutative C*-algebra. The Gelfand Transform $\Gamma_A : A \to C_bMA$ is an isometric *-isomorphism.*

The theorem packs a lot of information into one sentence. We break the theorem into the following sections:

(1) Γ_A is a $*$-homomorphism.
(2) Γ_A is an isometry (and therefore injective).

(3) Γ_A is surjective.

The proofs in this section are taken from Davidson [1], chapters I.2 and I.3.

8.1. Γ **Is a $*$-homomorphism.** We already know that $\Gamma_A : A \to C_b MA$ is a unital Banach algebra homomorphism from our category theory arguments; $\Gamma_A \in Hom_{\boldsymbol{\mathcal{B}}\mathbf{an}}(A, C_b MA)$ by construction, hence it must have the properties of morphisms in the category $\boldsymbol{\mathcal{B}}\mathbf{an}$. The following proof that Γ_A is a $*$-homomorphism as long as A is a C*-algebra invokes the C*-identity.

Lemma 8.2. *If A is a unital commutative C*-algebra then Γ_A is a $*$-homomorphism.*

Proof. Let A be a C*algebra. Suppose first $a \in A$ is self-adjoint, i.e., $a = a^*$. Consider the following elements $U_t \in A$:

$$U_t = e^{ita} = \sum_{n \geq 0} \frac{(ita)^n}{n!},$$

where i is the complex unit and $t \in \mathbb{R}$. The infinite series is absolutely convergent in A, because $\|(ita)^n\| \leq t^n \|a\|^n$, and we know that the series

$$\sum_{n \geq 0} \frac{(t\|a\|)^n}{n!}$$

converges in $\mathbb{R}$. The adjoint map $a \mapsto a^*$ is continuous. Therefore the adjoint of U_t is

$$U_t^* = \sum_{n \geq 0} \frac{(\overline{it}a)^n}{n!} = \sum_{n \geq 0} \frac{(-ita)^n}{n!} = e^{-ita}.$$

We see that $U_t^* \cdot U_t = e^{ita} \cdot e^{-ita} = 1$. (The product rule for exponents is valid in this case, and can be proven directly using the power series.) Now we use the C*-identity,

$$\|U_t\|^2 = \|U_t^* U_t\| = \|1\|.$$

Note that $1^* = 1$ in any C*-algebra, as $1^* = (1 \cdot 1)^* = 1^* \cdot 1^*$. For the C*-identity, then, we have that $\|1\| = \|1^* 1\| = \|1\|^2$, and so $\|1\| = 1$, and $\|U_t\|^2 = \|1\|$. By Lemma 4.4 we know that a multiplicative linear functional $\varphi : A \to \mathbb{C}$ has norm $\|\varphi\| = 1$. Thus $|\varphi(U_t)| \leq \|\varphi\| \cdot \|U_t\| = 1$. Now, φ is continuous because it is bounded in the operator norm. Then the following construction gives that

$$\varphi(U_t) = \sum_{n \geq 0} \frac{(it\varphi(a))^n}{n!} = e^{it\varphi(a)}.$$

As a result,

$$|e^{it\varphi(a)}| \leq 1.$$

For any complex number $z = x + iy$, with $x, y \in \mathbb{R}$, we have $|e^z| = e^x$. As a result,

$$|e^{it\varphi(a)}| = e^{-t \cdot Im\varphi(a)} \leq 1.$$

But unless $Im(\varphi(a)) = 0$ we can always choose t such that $e^{-t \cdot Im\varphi(a)} > 1$. For example, pick $t = -Im\varphi(a)$. Therefore $\varphi(a)$ must be a real number for any self-adjoint $a \in A$ and $\varphi \in M(A)$. As a result, the Gelfand transform $\Gamma_A(a)(\varphi) = \varphi(a)$ is real for every φ. Therefore,

$$a = a^* \implies \Gamma_A(a)^* = \Gamma_A(a).$$

Now let $a \in A$ be an arbitrary element. Every element a in a Banach*-algebra is a linear combination $a = b + ic$ of self-adjoint elements b and c as follows: $b = \frac{a+a^*}{2}$ and $c = \frac{a-a^*}{2i}$. So

$$\Gamma_A(a^*) = \Gamma_A((b + ic)^*) = \Gamma_A(b - ic) = \Gamma_A(b) - i\Gamma_A(c),$$

since Γ_A is linear. Then

$$\Gamma_A(b) - i\Gamma_A(c) = (\Gamma_A(b) + i\Gamma_A(c))^* = \Gamma_A(b + ic)^* = \Gamma_A(a)^*.$$

□

8.2. Γ **Is an Isometry.** In order to prove that the Gelfand Transform is an isometry, we need some facts from the spectral theory of unital Banach algebras.

Definition 8.1. Let A be a unital Banach algebra. Then for $a \in A$, the **spectrum** of a is the subset of $\mathbb{C}$

$$\operatorname{spec}(a) = \{\lambda \in \mathbb{C} : a - \lambda \cdot I_A \text{ is not invertible in } A\}.$$

The **spectral radius** of a is

$$\operatorname{spr}(a) = \sup\{|\lambda| : \lambda \in \operatorname{spec}(a)\}.$$

It can be shown that $\operatorname{spec}(a)$ is a non-empty compact set. Thus $\operatorname{spr}(a)$ is well-defined and finite. We need the following result of Beurling, the proof of which can be found in Davidson [1], Proposition I.2.3.

Fact (Beurling). Let A be a unital Banach algebra. Then for every $a \in A$,

$$\operatorname{spr}(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$$

For abelian Banach algebras, the spectral radius is closely related to the Gelfand transform. First, we note the following lemmas.

Lemma 8.3. *For $f \in C_b X$, $\operatorname{spec}(f) = \operatorname{range}(f)$.*

Proof. A function g is not invertible in C_bX if there exists $x_0 \in X$ such that $g(x_0) = 0$. (Note that we consider the inverse in the Banach algebra.) Then $f - \lambda \cdot I$ is not invertible for precisely all λ such that $f(x) = \lambda$. Then it is clear that $\operatorname{spec}(f) = (range)(f)$. $\square$

Corollary 8.4. *For $f \in C_bX$, $\operatorname{spr}(f) = \|f\|_\infty$.*

Lemma 8.5. *Let A be a unital abelian Banach algebra. Then for every $a \in A$,* $\operatorname{spec}(a) = \operatorname{spec}(\Gamma_A(a))$.

Proof. First, $\operatorname{spec}(\Gamma_A)(a) \subseteq \operatorname{spec}(a)$. We prove the contrapositive; suppose $\lambda \notin \operatorname{spec}(a)$. We will then argue that $\lambda \notin \operatorname{spec}(\Gamma_A(a))$. If $\lambda \notin \operatorname{spec}(a)$, then there exists some $b \in A$ such that $b \cdot (a - \lambda \cdot 1_A) = 1_A$. In that case, for every multiplicative linear functional $\varphi : A \to \mathbb{C}$, $\varphi(b \cdot (a - \lambda \cdot 1_A)) = \varphi(b) \cdot \varphi(a - \lambda \cdot 1_A) = 1$ in which case it cannot be true that $\varphi(a - \lambda \cdot 1_A) = 0$. Therefore $\varphi(a) - \lambda\varphi(1_A) \neq 0$ so $\varphi(a) \neq \lambda$. Therefore, for every φ, $\varphi(a) \neq \lambda$, so $\lambda \notin \operatorname{range}(\Gamma_A(a))$, and therefore $\lambda \notin \operatorname{spec}(\Gamma_A(a))$.

Now, we argue that $\operatorname{spec}(a) \subseteq \operatorname{spec}(\Gamma_A)(a)$. Let $\lambda \in \operatorname{spec}(a)$. In that case, $a - \lambda \cdot 1_A$ is not invertible, which means that there does not exist any b such that $(a - \lambda \cdot 1_A) \cdot b = 1_A$; therefore, we know that the ideal

$$J = \{(a - \lambda \cdot 1_A) \cdot b : b \in A\}$$

must not contain 1_A and is therefore proper. Suppose $J \subseteq K$, where K is any maximal ideal. In that case, there is a corresponding multiplicative linear functional φ with kernel K such that $\varphi(a - \lambda \cdot 1_A) = 0$, by definition. This gives us that $\varphi(a) - \lambda = 0$, so there exists $\varphi \in MA$ such that $\varphi(a) = \lambda$. As a result, we know that $\lambda \in \operatorname{range}(\Gamma_A(a)) = \{\varphi(a) : \varphi \in MA\}$. In that case, we know that $\lambda \in \operatorname{spec}(\Gamma_A(a))$. $\square$

We derive the following corollary, combining the results above.

Corollary 8.6. *Let A be a unital abelian Banach algebra. Then for every $a \in A$,* $\operatorname{spr}(a) = \|\Gamma_A(a)\|_\infty$, *and* $\operatorname{spec}(a) = \{\varphi(a) : \varphi \in MA\}$.

Now we have enough background to prove our objective.

Theorem 8.7. *Let A be a unital commutative C*-algebra. The Gelfand transform $\Gamma_A : A \to C_bM(A)$ is an isometry, i.e.*

$$\|\Gamma_A(a)\|_\infty = \|a\| \qquad \text{for all } a \in A.$$

Proof. We first prove that the theorem holds for a self-adjoint element $a \in A$. If $a = a^*$ the C*-identity gives $\|a\|^2 = \|a^*a\| = \|a^2\|$, and by induction we get $\|a^{(2^n)}\| = \|a\|^{2^n}$. Then Beurling's equality gives

$$\operatorname{spr}(a) = \lim_{n\to\infty} \|a^{(2^n)}\|^{\frac{1}{2^n}} = \lim_{n\to\infty} (\|a\|^{2n})^{\frac{1}{2^n}} = \|a\|.$$

But since $\mathrm{spr}(a) = \|\Gamma_A(a)\|_\infty$ we get $\|\Gamma_A(a)\|_\infty = \|a\|$.

Now, we use the self-adjoint case to prove the theorem holds generally. If $a \in A$ is an arbitrary element, then $(a^*a)^* = a^*(a^*)^* = a^*a$, and so we have

$$\|a^*a\| = \|\Gamma_A(a^*a)\|_\infty.$$

Since Γ_A is a $*$-homomorphism,

$$\|\Gamma_A(a^*a)\|_\infty = \|\Gamma_A(a)^*\Gamma_A(a)\|_\infty = \|\Gamma_A(a)\|_\infty^2.$$

Now we can invoke the C*-identity again to see that $\|a^*a| = \|a\|^2$, in which case we have shown that $\|\Gamma_A(a)\|_\infty^2 = \|a\|^2$, and therefore $\|\Gamma_A(a)\| = \|a\|$ for any $a \in A$ so Γ_A is an isometry. □

Corollary 8.8. *Let A be a unital commutative C*-algebra. Then the Gelfand transform $\Gamma_A : A \to C_bM(A)$ is an injective map.*

Proof. We repeat the well-known result that isometries are injective in our context. Take $\Gamma_A(a) = \Gamma_A(b)$. Then for every $\varphi \in MA$, $\varphi(a) = \varphi(b)$, so we plainly have that

$$\|\Gamma_A(a) - \Gamma_B(b)\|_\infty = 0 = \|\Gamma_A(a-b)\|_\infty,$$

so $\|a-b\|_A = 0$, and then $a = b$. □

8.3. Γ **Is Surjective.** Surjectivity of the Gelfand transform for C*-algebras follows from the Stone-Weierstrass theorem.

Lemma 8.9. *Let A be a unital commutative C*-algebra. The Gelfand transform Γ_A is surjective.*

Proof. Because Γ_A is a $*$-homomorphism, the range $\Gamma_A(A)$ is a unital $*$-subalgebra of C_bMA. Because Γ_A is an isometry and A is complete, $\Gamma_A(A)$ is also complete and hence closed in C_bMA.

We now invoke the Stone-Weierstrass theorem.

Fact (Stone-Weierstrauss). Let X be a compact Hausdorff space. Every unital $*$-subalgebra of $C(X)$ that separates points in X is dense in $C(X)$. (See Rudin [7], p. 121-122.)

The fact that $\Gamma_A(A)$ separates points in MA is trivial. Given $\varphi \neq \psi$ in MA, there is $a \in A$ with $\varphi(a) \neq \psi(a)$. But that means that $\Gamma_A(a)(\varphi) \neq \Gamma_A(a)(\psi)$, so $\Gamma_A(a)$ separates φ and ψ. Thus, by the Stone-Weierstrass Theorem $\Gamma_A(A)$ is dense in C_bMA. Therefore $\overline{\Gamma_A(A)} = C_bMA$, but $\overline{\Gamma_A(A)} = \Gamma_A(A)$, as it is closed. Therefore, $\Gamma_A(A) = C_bMA$, in which case for every $\hat{a} \in C_bMA$, there exists $a \in A$ such that $\Gamma_A(a) = \hat{a}$, and the function Γ_A is surjective. □

References

[1] Davidson, Kenneth R. *C^*-Algebras by Example.* Providence, Rhode Island: American Mathematical Society, 1996.

[2] Gelfand, Israel, and Mark Naimark. "On the imbedding of normed rings into the ring of operators in Hilbert space." *Recueil Mathematique* 2.12(54) (1943): 197–217.

[3] Herstein, I. N. *Abstract Algebra.* Hoboken, NJ: John Wiley & Sons, 1990.

[4] Maclane, Saunders. *Categories for the Working Mathematician.* New York: Springer-Verlag, 1998.

[5] Munkres, James. *Topology: A First Course.* Englewood Cliffs, NJ: Prentice-Hall, Inc., 1975.

[6] Rosenlicht, Maxwell. *Introduction to Analysis.* New York: Dover Publications, 1986.

[7] Rudin, Walter. *Functional Analysis.* New York: McGraw-Hill, 1973.