SPECIAL VALUES OF RIEMANN ZETA FUNCTION

A Thesis
Submitted to the Department of Mathematics

By

MILAN P. HUYNH

Dartmouth College
Hanover, NH

May 2017

Advisor:

NAOMI TANABE

ABSTRACT

The Riemann zeta function is defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} (1 - p^{-s})^{-1}$ for complex numbers s with $\Re(s) > 1$, and it can be analytically continued to the whole complex plane $\mathbb{C}$ except at $s = 1$. Furthermore, the values of $\zeta(2k)$ at positive integers k can be described in terms of Bernoulli numbers and π^{2k}. This note details the properties of $\zeta(s)$ and Bernoulli numbers, delineates a newly published method for evaluating $\zeta(2k)$, and summarizes properties of Dirichlet L-functions.

ACKNOWLEDGEMENTS

First and foremost, I sincerely thank my thesis advisor, Professor Naomi Tanabe, for her immense patience, mentorship, and motivation on this project. I am extremely grateful for my professors at the Department of Mathematics for their support these four years. Also, I acknowledge Professor Dana Williams for his thoughts.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

This note gives a robust understanding of the Riemann zeta function and a connection to a generalization of it.

In the mid-seventeenth century, Leonhard Euler first encountered the Riemann zeta function (known then as the "zeta function") as the following series for a natural number s,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

when he solved the Basel Problem that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Euler later showed that $\zeta(s)$ has the identity

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \left(1 - p^{-s}\right)^{-1},$$

where P is the set of all primes, thus relating the Riemann zeta function to prime numbers. Although Chebyshev showed that the series representation of $\zeta(s)$ holds only for complex numbers s with $\Re(s) > 1$, Bernhard Riemann further demonstrated that it accepts an analytic continuation to the whole complex plane with a simple pole at $s = 1$ and satisfies the functional equation. Chapters 2 and 3 detail this history.

In addition, we can evaluate "special values" of $\zeta(2k)$ for positive integers k. More precisely, if B_{2k} denotes the $2k$-th Bernoulli number, then

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!} B_{2k}.$$

Chapter 4 establishes properties for Bernoulli numbers and polynomials. Two ways of evaluating $\zeta(2k)$ are explained in Chapter 5. In particular, Section 5.1 describes the classical method of evaluating $\zeta(2k)$ using complex analysis, and Section 5.2 details a new method, which was shown by Ciaurri, Navas, Ruiz, and Varona in their recent paper [11], that uses only properties of Bernoulli polynomials and telescoping series.

One way to generalize the Riemann zeta function is to "twist" each term of the series by a nice sequence. In 1837, Johann Dirichlet first defined the Dirichlet L-function by assigning a Dirichlet character χ to the numerator of the series

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}.$$

Chapter 6 summarizes properties of Dirichlet L-functions that are analogous to properties of the Riemann zeta function. Our main goal is to generalize some of the method introduced in [11] to the setting of Dirichlet L-functions. It is still in progress, so it is not included in this submission.

CHAPTER 2
RIEMANN ZETA FUNCTION

The Riemann zeta function is a famous function that was introduced as a series representation of the form $\sum_{n=1}^{\infty} \frac{1}{n^k}$ with a natural number k. Leonhard Euler observed its deep connection to the prime numbers and evaluated its values for some positive integers k. Later on, Bernhard Riemann extended it as a function on the whole complex plane (except at 1). The aim of this chapter is to explain some of its important properties and to state its analytic continuation and special values, which will be described later. This expository note primarily references [17], [2], and [22].

Definition 2.0.1 *Let s be a complex number with $\Re(s) > 1$. We define the* ***Riemann zeta function*** *$\zeta(s)$ as the following series:*

$$\zeta(s) \coloneqq \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

To further proceed, we introduce the definition of absolute convergence, which is crucial to our study. An infinite series $\sum_{i=1}^{\infty} a_i$ is said to be **absolutely convergent** if there exists L in $\mathbb{R}_{\geq 0}$ such that $\sum_{i=1}^{\infty} |a_i| = L$. It is a well-known property in elementary analysis that an absolutely convergent series converges to the same value when terms are rearranged.

If k is a real number greater than 1, it is known that $\sum_{n=1}^{\infty} \frac{1}{n^k}$ converges absolutely, which follows from an application of the integral test. Indeed, it is observed that, if $k > 1$ is real,

$$\sum_{n=1}^{\infty} \left| \frac{1}{n^k} \right| = \sum_{n=1}^{\infty} \frac{1}{n^k} \leq 1 + \int_1^{\infty} \frac{1}{x^k} \mathrm{d}x = 1 + \frac{1}{k-1}.$$

We now aim to extend this property for any complex numbers s with $\Re(s) > 1$. See the following proposition.

Proposition 2.0.2 *$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for $\Re(s) > 1$.*

Proof. Let $s = x + iy$ with $x > 1$. Then $n^s = n^x n^{iy} = n^x e^{iy \log(n)}$, and therefore $|n^s| = \left|n^x\right|\left|e^{iy \log(n)}\right| = n^x \cdot 1$. Hence,

$$\sum_{n=1}^{\infty} \frac{1}{\left|n^{(x+iy)}\right|} = \sum_{n=1}^{\infty} \frac{1}{n^x} = \zeta(x).$$

Since $\zeta(x)$ is absolutely convergent, the result follows.

Another interesting (and very important) property of $\zeta(s)$ is a connection with all prime numbers, which was first proved by Euler. In what follows, we show interesting properties of prime numbers using the "Euler product" and elementary number theory.

Proposition 2.0.3 (Euler Product of the Riemann Zeta Function) *The Riemann zeta function can be written as a product of prime factors, called the Euler product. Let s be a complex number with $\Re(s) > 1$, and let P be the set of prime numbers. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in P} \frac{1}{1-p^{-s}}. \tag{2.0.1}$$

Proof. There are two easy ways to see this. First, consider the right hand side, since $p^{-s} < 1$, then we can write $\frac{1}{1-p^{-s}}$ as a geometric series. That is,

$$\begin{aligned}\prod_{p \in P} \frac{1}{1-p^{-s}} &= \prod_{p \in P} \sum_{k=0}^{\infty} p^{-ks} = \prod_{p \in P} \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots \right), \\ &= 1 + \sum_{a=0}^{\infty} \sum_{p \in P} \frac{1}{p^{as}} + \sum_{a_1, a_2 = 0}^{\infty} \sum_{p_1, p_2 \in P} \frac{1}{p_1^{a_1 s} p_2^{a_2 s}} + \cdots .\end{aligned}$$

By the unique factorization theorem, we have

$$\prod_{p \in P} \frac{1}{1-p^{-s}} = \zeta(s).$$

For the second method, let us first consider the following: since

$$\frac{1}{2^s}\zeta(s) = \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots ,$$

we observe that

$$\zeta(s) - \frac{1}{2^s}\zeta(s) = \left(1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + \cdots \right) - \left(\frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{6^s} + \frac{1}{8^s} + \frac{1}{10^s} + \cdots \right).$$

Since $\zeta(s)$ is absolutely convergent when $\Re(s) > 1$, we may interchange terms accordingly, and take out terms with even denominators, that is,

$$\zeta(s)\left(1 - \frac{1}{2^s}\right) = 1 + \frac{1}{3^s} + \frac{1}{5^s} + \frac{1}{7^s} + \frac{1}{9^s} + \cdots = \sum_{\substack{n=1 \\ 2 \nmid n}}^{\infty} \frac{1}{n^s}.$$

Now consider multiplying the equation by $1 - \frac{1}{3^s}$, which gives

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right) = \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} \frac{1}{n^s} - \sum_{\substack{n=1 \\ 2\nmid n}}^{\infty} \frac{1}{(3n)^s} = \sum_{\substack{n=1 \\ 2\nmid n \\ 3\nmid n}}^{\infty} \frac{1}{n^s}.$$

Inductively, we have

$$\zeta(s)\left(1-\frac{1}{2^s}\right)\left(1-\frac{1}{3^s}\right)\left(1-\frac{1}{5^s}\right)\cdots = \zeta(s)\prod_{p\in P}\left(1-\frac{1}{p^s}\right) = \sum_{\substack{n=1 \\ \forall p\in P,\ p\nmid n}}^{\infty} \frac{1}{n^s} = 1.$$

Thus we get our desired result. One direct result we can see from the Euler product is the infinitude of prime numbers.

Corollary 2.0.4 *There are infinitely many primes.*

Proof. Suppose there are finitely many prime numbers. Then $|P|$ is finite and so $\zeta(1) = \prod_{p\in P}\left(1-\frac{1}{p}\right)^{-1}$ would converge. It contradicts because it is known that $\zeta(1) = \sum_{n=1}^{\infty}\frac{1}{n}$ diverges. Further observation of $\zeta(s)$ provides more information on prime numbers. For instance, we can show that the number of prime numbers is greater than the number of square numbers. This can be viewed by the following steps. First take the logarithm of both sides of Equation (2.0.1) and see

$$\log\zeta(s) = \log\left(\prod_{p\in P}\frac{1}{1-p^{-s}}\right) = \sum_{p\in P}\log\left(\frac{1}{1-p^{-s}}\right).$$

We note that the Taylor expansion of $\log\frac{1}{1-x}$ is $\sum_{n=1}^{\infty}\frac{x^n}{n}$ for $|x| < 1$. Since $\left|\frac{1}{1-p^{-s}}\right| < 1$ if $\Re(s) > 1$, we may write

$$\log\zeta(s) = \sum_{p\in P}\sum_{n=1}^{\infty}\frac{\left(p^{-s}\right)^n}{n} = \sum_{p\in P}\sum_{n=1}^{\infty}\frac{1}{np^{ns}}.$$

Let us now put $P(s) = \sum_{p\in P}\frac{1}{p^s}$ so that $\log\zeta(s)$ can be written as

$$\log\zeta(s) = \sum_{p\in P}\frac{1}{p^s} + \sum_{p\in P}\sum_{n=2}^{\infty}\frac{1}{np^{ns}} = P(s) + \sum_{p\in P}\sum_{n=2}^{\infty}\frac{1}{np^{ns}}.$$

We note that $P(s)$ is called the prime zeta function. We now direct our attention to the last term of the above equations. Since $\sum_{n=2}^{\infty}\frac{1}{np^{ns}}$ is absolutely convergent, we may interchange the summations and obtain

$$\log\zeta(s) = P(s) + \sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{ns}}. \tag{2.0.2}$$

Let us write $s = x + iy$. Since $|e^{iy}| = 1$ for any $y \in \mathbb{R}$,

$$\left|\sum_{p\in P}\frac{1}{p^{ns}}\right| \le \sum_{p\in P}\left|\frac{1}{p^{ns}}\right| = \sum_{p\in P}\frac{1}{|p^{nx+iyn}|} = \sum_{p\in P}\frac{1}{|p^{nx}||p^{nyi}|}\sum_{p\in P}\frac{1}{p^{nx}|e^{iyn\log(p)}|} = \sum_{p\in P}\frac{1}{p^{nx}}.$$

Thus, for $x > 1$, we have

$$\left|\sum_{p\in P}\frac{1}{p^{ns}}\right| \le \left|\frac{1}{p^{nx}}\right| \le \sum_{p\in P}\frac{1}{p^{n}} \le \sum_{j=2}^{\infty}\frac{1}{j^{n}}.$$

Now we obtain an upper bound of the right hand side as

$$\sum_{j=2}^{\infty}\frac{1}{j^{n}} < \sum_{j=2}^{\infty}\int_{j-1}^{j}\frac{1}{t^{n}}\mathrm{d}t = \int_{1}^{\infty}\frac{1}{t^{n}}\mathrm{d}t = \frac{1}{n-1},$$

which transitively means

$$\left|\sum_{p\in P}\frac{1}{p^{ns}}\right| \le \frac{1}{n-1}.$$

Thus, we obtain the following upper bound,

$$\left|\sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{ns}}\right| \le \sum_{n=2}^{\infty}\frac{1}{n}\left|\sum_{p\in P}\frac{1}{p^{ns}}\right| \le \sum_{n=2}^{\infty}\frac{1}{n}\cdot\frac{1}{n-1} = 1.$$

The last equality uses the fact that the partial sum $\sum_{n=2}^{M}\frac{1}{n(n-1)} = \frac{M-1}{M}$. In particular,

$$\left|\sum_{n=2}^{\infty}\frac{1}{n}\sum_{p\in P}\frac{1}{p^{ns}}\right| \le 1$$

as $s \to 1$. Therefore, by Equation (2.0.1), it must be that $P(1) = \sum_{p\in P}\frac{1}{p}$ diverges since $\lim_{s\to 1^{+}}\zeta(s) = \infty$. As we will see later in Section 5.1, $\zeta(2) = \frac{\pi^2}{6}$, which implies that, although there are infinitely many primes and squares, the primes are "more numerous" than the squares. More details on this topic can be found in, for example, [14].

Another important property of Riemann zeta function is to have an analytic continuation to the complex plane (with a simple pole at $s = 1$), which was first proved by Bernhard Riemann in his 1859 manuscript (translated into English in [23]). Furthermore, it is known to satisfy the functional equation of the form

$$\zeta(s)\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s),$$

where $\Gamma(s)$ is the gamma function. Detail on analytic continuation and the functional equation is discussed in Chapter 3.

One of the interesting topics concerning the Riemann zeta function is to evaluate so-called "special values," that is to study the values of $\zeta(s)$ at $s = 2k$ with $k \in \mathbb{N}$. It is one of our main goals to fully understand the following theorem.

Theorem 2.0.5 *Let k be a natural number and B_{2k} denote the $2k$-th Bernoulli number. Then*

$$\zeta(2k) = \frac{(-1)^{k+1}(2\pi)^{2k}}{2(2k)!}B_{2k}.$$

Bernoulli numbers are defined by using their recursive definition as $\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0$, where $k > 1$ and $B_0 = 1$. Their properties are explained in Chapter 4, and the proofs for Theorem 2.0.5 will be shown in Sections 5.1 and 5.2.

CHAPTER 3
ANALYTIC CONTINUATION

Analytic continuation of a holomorphic function is a process of extending the function's domain to a larger domain. See the definition below as described in Section 5.8 of [19].

Definition 3.0.1 (Analytic Continuation) *Let f be analytic in domain D_1 and g be analytic in domain D_2. If $D_1 \cap D_2$ is nonempty and $f(z) = g(z)$ for all z in $D_1 \cap D_2$, then we call g a **direct analytic continuation** of f to D_2.*

Using a direct analytic continuation of a function f we can define a new function on an extended domain as follows.

Theorem 3.0.2 *Given an analytic function f in domain D_1 and a direct analytic continuation, g, of f to domain D_2, we can form the function*

$$F(z) := \begin{cases} f(z) & \text{for } z \in D_1 \\ g(z) & \text{for } z \in D_2, \end{cases}$$

which is analytic on $D_1 \cup D_2$.

The uniqueness of such function can be shown by studying the Taylor expansion of f and g. The goal of this section is to study the analytic continuation of $\zeta(s)$. This section is mainly studied from Chapter 2 of [1], [3], [6], and [2].

3.1 Gamma Function

We start with analytic continuation of the gamma function, which helps in proving the analytic continuation of the Riemann zeta function later.

Definition 3.1.1 *Let s be a complex number with $\Re(s) > 0$. Then we define the **gamma function** as*

$$\Gamma(s) = \int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t}.$$

It is noted that $\int_0^\infty e^{-t} t^s \frac{\mathrm{d}t}{t} < \infty$ for $\Re(s) > 0$. In particular, we easily observe by definition that $\Gamma(1) = \int_0^\infty e^{-t} t^1 \frac{\mathrm{d}t}{t} = -e^{-t}|_0^\infty = 1$ and that $\Gamma(x)$ is real for all real numbers x. Moreover, we have the following proposition.

Proposition 3.1.2 *Let s be a complex number with $\Re(s) > 0$. Then*

$$\Gamma(s+1) = s\Gamma(s).$$

Proof. Applying integration by parts to

$$\Gamma(s+1) = \int_0^\infty e^{-t} t^{s+1} \frac{\mathrm{d}t}{t},$$

we see that

$$\Gamma(s+1) = -e^{-t} t^s \Big|_{t=0}^{t=\infty} - \int_0^\infty -e^{-t} s t^s \frac{\mathrm{d}t}{t} = 0 + s\Gamma(s).$$

□

The following corollary follows immediately from Proposition 3.1.2.

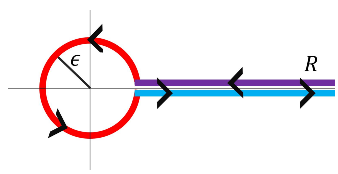
Corollary 3.1.3 *For any positive integer n,*

$$\Gamma(n) = (n-1)!$$

We can use Proposition 3.1.2 to define analytic continuation for the $\Gamma(s)$ where $\Re(s) \leq 0$. Specifically, we observe that the right hand side of $\Gamma(s) = \frac{\Gamma(s+1)}{s}$ is analytic when $\Re(s) > -1$ with a simple pole at $s = 0$ with residue 1. If we repeat this extension, we can continue extending $\Gamma(s)$ to the whole complex plane with simple poles when s is a non-positive integer.

There is another way to define analytic continuation of the Gamma function, which is to use contour integration. In the following proposition, we introduce this method.

Proposition 3.1.4 (Analytic Continuation of the Gamma Function) *Let the contour C be the Hankel Curve shown below.*



Then the gamma function $\Gamma(s)$ can be represented as the following integral where the branch cut is real non-negative axis $x \geq 0$:

$$\Gamma(s) = \frac{1}{e^{2\pi i s} - 1} \oint_C t^{s-1} e^{-t} \mathrm{d}t.$$

Due to the branch cut and the denominator of $e^{2\pi i s} - 1$, this representation is defined for complex numbers s where $s \notin \mathbb{R}_{\geq 0} \cup \mathbb{Z}_{<0}$.

Proof. Since C is a contour on the complex plane, we integrate over $t = x + iy$ and consider 3 parts of the Hankel Curve C as

$$\oint_C t^{s-1} e^{-t} \mathrm{d}t, \qquad = \int_{\text{purple line}} t^{s-1} e^{-t} \mathrm{d}t + \int_{\text{red circle}} t^{s-1} e^{-t} \mathrm{d}t + \int_{\text{blue line}} t^{s-1} e^{-t} \mathrm{d}t.$$

The purple line is a line segment for x from R to ϵ (on the positive imaginary side) with an arbitrarily small positive number ϵ. Therefore, we write the integral as

$$\int_{\text{purple line}} t^{s-1} e^{-t} \mathrm{d}t = \int_R^\epsilon x^{s-1} e^{-x} \mathrm{d}x.$$

For the red curve, write t in polar coordinates, that is, $t = \epsilon e^{i\theta}$ with $\theta \in (0, 2\pi)$, and so $\frac{\mathrm{d}t}{\mathrm{d}\theta} = i\epsilon e^{i\theta}$. Hence,

$$\int_{\text{red circle}} t^{s-1}e^{-t}\mathrm{d}t = \int_0^{2\pi} \left(\epsilon e^{i\theta}\right)^{s-1} e^{-\epsilon e^{i\theta}} i\epsilon e^{i\theta}\mathrm{d}\theta.$$

The blue line is a line segment for x from ϵ to R (on the negative imaginary side). Since it is a contour after wrapping around the origin, the phase angle is now $2\pi i$. Therefore, writing $t = xe^{2\pi i}$, we have

$$\int_{\text{blue line}} t^{s-1}e^{-t}\mathrm{d}t = \int_\epsilon^R \left(xe^{2\pi i}\right)^{s-1} e^{-xe^{2\pi i}}\mathrm{d}x.$$

Hence, we have

$$\begin{aligned}
&\lim_{R\to\infty,\epsilon\to 0} \oint_C t^{s-1}e^{-t}\mathrm{d}t \\
&\quad= \lim_{R\to\infty,\epsilon\to 0} \left(\int_R^\epsilon x^{s-1}e^{-x}\mathrm{d}x + \int_0^{2\pi} \left(\epsilon e^{i\theta}\right)^{s-1} e^{-\epsilon e^{i\theta}} i\epsilon e^{i\theta}\mathrm{d}\theta + \int_\epsilon^R \left(xe^{2\pi i}\right)^{s-1} e^{-xe^{2\pi i}}\mathrm{d}x\right) \\
&\quad= \lim_{R\to\infty,\epsilon\to 0} \left(-\int_\epsilon^R x^{s-1}e^{-x}\mathrm{d}x + i\epsilon \int_0^{2\pi} \left(\epsilon e^{i\theta}\right)^{s-1} e^{-\epsilon e^{i\theta}} e^{i\theta}\mathrm{d}\theta + e^{2\pi i(s-1)} \int_\epsilon^R x^{s-1}e^{-x}\mathrm{d}x\right) \\
&\quad= \lim_{R\to\infty} \left(-\int_0^R x^{s-1}e^{-x}\mathrm{d}x + 0 + e^{2\pi is}e^{-2\pi i} \int_0^R x^{s-1}e^{-x}\mathrm{d}x\right) \\
&\quad= \lim_{R\to\infty} \left(-\int_0^R x^{s-1}e^{-x}\mathrm{d}x + e^{2\pi is}e^{-2\pi i} \int_0^R x^{s-1}e^{-x}\mathrm{d}x\right) \\
&\quad= \lim_{R\to\infty} \left(-\int_0^R x^{s-1}e^{-x}\mathrm{d}x + e^{2\pi is}(1) \int_0^R x^{s-1}e^{-x}\mathrm{d}x\right) \\
&\quad= \lim_{R\to\infty} \left(\left(-1 + e^{2\pi is}\right) \int_0^R x^{s-1}e^{-x}\mathrm{d}x\right) \\
&\quad= \left(e^{2\pi is} - 1\right) \int_0^\infty x^{s-1}e^{-x}\mathrm{d}x \\
&\quad= \left(e^{2\pi is} - 1\right) \Gamma(s).
\end{aligned}$$

In conclusion,

$$\oint_C t^{s-1}e^{-t}\mathrm{d}t = \left(e^{2\pi is} - 1\right)\Gamma(s), \tag{3.1.1}$$

which gives us our desired result of

$$\Gamma(s) = \frac{1}{e^{2\pi is} - 1} \oint_C t^{s-1}e^{-t}\mathrm{d}t.$$

□

Proposition 3.1.5 *The gamma function $\Gamma(s)$, as a function on $\mathbb{C}$, has simple poles at non-positive integers, $s = -n$, with residues $\frac{(-1)^n}{n!}$.*

Proof. We already showed that $\Gamma(s)$ has simple poles at non-positive integers. This can also be observed from appearance of the factor $(e^{2\pi is} - 1)^{-1}$ in the Hankel contour representation. To find the residues of $\Gamma(s)$ at negative integers $s = -n$, we first observe that if s was an integer, then $\oint_C t^{s-1}e^{-t}\mathrm{d}t$ would not require a branch cut. Therefore, $\oint_C t^{s-1}e^{-t}\mathrm{d}t$ becomes an integral over a circle around the origin, which is the red contour in the previous picture. Thus, we have

$$\oint_C t^{-n-1}e^{-t}\mathrm{d}t = \oint_{|t|=\epsilon} t^{-n-1}e^{-t}\mathrm{d}t = \frac{2\pi i}{n!}$$

because of Cauchy Residue Theorem. Therefore the residue of $\Gamma(-n)$ is

$$\lim_{s\to -n}\left(\frac{s-(-n)}{e^{2\pi is}-1}\oint_C t^{-n-1}e^{-t}\mathrm{d}t\right) = \frac{2\pi i}{n!}\lim_{s\to -n}\left(\frac{s+n}{e^{2\pi is}-1}\right) = \frac{(-1)^n}{n!}.$$

□

Another important property of the gamma function $\Gamma(s)$ is to satisfy the following functional equation, which is often called the Euler reflection formula of the gamma function.

Theorem 3.1.6 *For complex number s, such that $s \notin \mathbb{Z}$,*

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Proof. The proof is omitted for now, but can be found in many books, such as [4] and [12]. □

An interesting result that follows from the theorem above is that the gamma function is never zero, and thus the reciprocal of the gamma function is entire. Hence, we have the following corollary.

Corollary 3.1.7 *The following equation is entire (analytic on the entire complex plane).*

$$\frac{1}{\Gamma(s)} = \frac{\sin(\pi s)}{\pi}\Gamma(1-s)$$

Proof. We know that $\Gamma(1-s)$ has simple poles at $s = 1, 2, 3, \ldots$ by Proposition 3.1.5, and $\sin(\pi s)$ has zeroes at $s = 0, -1, -2, \ldots$. Therefore the simple poles, which are poles of order 1, of $\Gamma(s)$ become removable singularities. □

Before we close our focus on the gamma function, let us point out Legendre's Duplication Formula that provides a relation between $\Gamma(2s)$ and $\Gamma(s)$. This discussion can be found, for example, in Chapter 3 (p. 24) of [3] or in Chapter 5 of [9]. They use the beta function (also known as *Euler's First Integral*) to prove this relation. We only state the formula without a proof.

Proposition 3.1.8 (Legendre's Duplication Formula) *For any complex number s, the following functional equation holds:*

$$\Gamma(2s) = \frac{2^{2s-1}}{\sqrt{\pi}}\Gamma(s)\Gamma\left(s+\frac{1}{2}\right).$$

3.2 Jacobi Theta Function

The next function to consider is the Jacobi theta function, and (inadvertently) the Poisson summation formula. We begin by explaining the Jacobi theta function, and how Fourier analysis gives us relevant properties.

Definition 3.2.1 *For any complex number s, the **Jacobi theta function** $\theta(s)$ is defined as*

$$\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s}.$$

Note that since $-\pi n^2 s = -\pi(-n)^2 s$, we may write the Jacobi theta function as

$$\theta(s) = 1 + 2\sum_{n=1}^{\infty} e^{-\pi n^2 s}.$$

The summation $\sum_{n=1}^{\infty} e^{-\pi n^2 s}$ is often called the psi function and is denoted as $\Psi(s)$. Therefore, we may also write the Jacobi theta function as $\theta(s) = 1 + 2\Psi(s)$.

To prove the analytic continuation of Riemann zeta function, the important property needed of the theta function is its functional equation, which is stated below.

Proposition 3.2.2 *The functional equation of Jacobi theta function is given as*

$$\theta(s) = \frac{1}{\sqrt{s}}\theta\left(\frac{1}{s}\right). \tag{3.2.1}$$

In order to prove the proposition above, we first introduce the Poisson summation formula, following from [5].

Proposition 3.2.3 (Poisson Summation Formula) *Let f be a piece-wise continuous function defined on $\mathbb{R}$, satisfying the following condition: For all $c \in \mathbb{R}$,*

$$f(c) = \frac{1}{2}\left[\lim_{x\to c^-} f(x) + \lim_{x\to c^+} f(x)\right]$$

and $|f(c)| < a$ for some positive constant a. (This means that f is bounded.) Then its Poisson summation is

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} f(x)e^{-2\pi ikx}\mathrm{d}x.$$

The Poisson summation formula is easily derived from the Fourier series. See, for example, [5] and Chapter 2 of [8] for more details.

Proof for Proposition 3.2.2. While this proposition holds for $s \in \mathbb{C}$, we only prove the case of $s \in \mathbb{R}$. This proof can can be extended to the complex plane. Let $f(n) = e^{-\pi n^2 s}$ for a fixed $s \in \mathbb{R}$. Then applying the Poisson summation formula to $\theta(s)$, we see

$$\begin{aligned}\theta(s) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 s} &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi x^2 s} e^{-2\pi i k x} \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi x^2 s - 2\pi i k x} \mathrm{d}x.\end{aligned}$$

We use the Gauss integral trick to solve the integral above. Thus, we aim to form an integral that looks like e^{-a^2} by completing the square for $-\pi x^2 s - 2\pi i k x$. Indeed, we may write

$$\begin{aligned}\theta(s) &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi x^2 s - 2\pi i k x} \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi s(x^2 + 2i\frac{k}{s}x + i^2\frac{k^2}{s^2} - i^2\frac{k^2}{s^2})} \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi s((x+i\frac{k}{s})^2 - i^2\frac{k^2}{s^2})} \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\pi s(x+i\frac{k}{s})^2 + \pi s(-1)\frac{k^2}{s^2})} \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} \int_{-\infty}^{\infty} e^{-\frac{\pi k^2}{s}} e^{-\pi s(x+i\frac{k}{s})^2} \mathrm{d}x \\ &= \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s(x+i\frac{k}{s})^2} \mathrm{d}x.\end{aligned}$$

We now want to introduce a substitution, $x + i\frac{k}{s} = a$ Since $i\frac{k}{s}$ is a constant, $\mathrm{d}x = \mathrm{d}a$. Therefore, we have

$$\theta(s) = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s a^2} \mathrm{d}a,$$

which is shown in Section 5.1 of [16]. Now we have the form to use the Gauss Integral trick,

$\int_{-\infty}^{\infty} e^{-bx^2}\mathrm{d}x = \sqrt{\frac{\pi}{b}}$ for positive b, and therefore

$$\theta(s) = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{s}} \int_{-\infty}^{\infty} e^{-\pi s a^2}\mathrm{d}a = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{s}} \sqrt{\frac{\pi}{\pi s}} = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{s}} \sqrt{\frac{\pi}{\pi s}} = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{s}} \sqrt{\frac{1}{s}}$$
$$= \frac{1}{\sqrt{s}}\theta\left(\frac{1}{s}\right).$$

This completes the proof. □

Applying Proposition 3.2.2 to $2\Psi(s) = \theta(s) - 1$, we observe the following:

Corollary 3.2.4

$$\Psi(s) = \frac{1}{\sqrt{s}}\Psi\left(\frac{1}{s}\right) + \frac{1}{2\sqrt{s}} - \frac{1}{2}.$$

This gives us enough tools to analytically extend the Riemann zeta function to $\mathbb{C}$. The following section is devoted to proving the analytic continuation of $\zeta(s)$ as well as its functional equation. This section is mainly adopted from [7] and [23], with more details.

3.3 Completed Riemann Zeta Function

Theorem 3.3.1 *The Riemann zeta function $\zeta(s)$ can be analytically continued to the whole complex plane except at $s = 1$.*

In order to prove Theorem 3.3.1, we first define the completed zeta function $\Lambda(s)$ and prove some of its properties.

Theorem 3.3.2 *Define the **completed zeta function** $\Lambda(s)$ as*

$$\Lambda(s) := \pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s)$$

for $\Re(s) > 1$. Then it can be analytically continued to the whole complex plane except when $s = 0$ and 1, and it satisfies the functional equation,

$$\Lambda(s) = \Lambda(1 - s).$$

Proof. We first note that $\Lambda(s)$ is well-defined where $\Re(s) > 1$ because $\zeta(s)$ and $\Gamma\left(\frac{s}{2}\right)$ are well-defined in that region. Furthermore, the integral representation of $\Gamma(\frac{s}{2})$ holds, so

$$\Gamma\left(\frac{s}{2}\right) = \int_0^{\infty} t^{\frac{s}{2}-1}e^{-t}\mathrm{d}t. \tag{3.3.1}$$

Let $n \in \mathbb{Z}_{>0}$ and substitute $t = \pi n^2 x$ into Equation (3.3.1). Then $\Gamma(\frac{s}{2})$ becomes

$$
\begin{aligned}
\Gamma\left(\frac{s}{2}\right) &= \int_0^\infty (\pi n^2 x)^{\frac{s}{2}-1} e^{-\pi n^2 x} \pi n^2 \mathrm{d}x \\
&= \int_0^\infty \pi^{\frac{s}{2}-1} n^{2(\frac{s}{2}-1)} x^{\frac{s}{2}-1} e^{-\pi n^2 x} \pi n^2 \mathrm{d}x \\
&= \pi^{\frac{s}{2}} n^s \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} \mathrm{d}x.
\end{aligned}
$$

Multiplying both sides of the equality by $\frac{\pi^{-\frac{s}{2}}}{n^s}$, we get

$$
\frac{\pi^{-\frac{s}{2}}}{n^s} \Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} \mathrm{d}x.
$$

Summing over positive integers n yields

$$
\sum_{n=1}^\infty \frac{\pi^{-\frac{s}{2}}}{n^s} \Gamma\left(\frac{s}{2}\right) = \sum_{n=1}^\infty \int_0^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} \mathrm{d}x.
$$

Notice that the left hand side of the above equation is $\pi^{-\frac{\pi}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s)$. For the right hand side, we can interchange the integral and summation because both are absolutely convergent for $\Re(s) > 1$. The above expression is then equal to

$$
\begin{aligned}
\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) &= \int_0^\infty \sum_{n=1}^\infty x^{\frac{s}{2}-1} e^{-\pi n^2 x} \mathrm{d}x \\
&= \int_0^\infty x^{\frac{s}{2}-1} \sum_{n=1}^\infty e^{-\pi n^2 x} \mathrm{d}x.
\end{aligned}
$$

It is observed that the summation on the right hand side of the above equation is nothing but the Ψ-function, and thus

$$
\Lambda(s) = \zeta(s) \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) = \int_0^\infty x^{\frac{s}{2}-1} \Psi(x) \mathrm{d}x.
$$

Now we split the right hand side into two integrals, namely

$$
\Lambda(s) = \int_0^1 x^{\frac{s}{2}-1} \Psi(x) \mathrm{d}x + \int_1^\infty x^{\frac{s}{2}-1} \Psi(x) \mathrm{d}x. \tag{3.3.2}
$$

We compute the first integral of Equation (3.3.2) by applying Corollary 3.2.4. Indeed,

$$\begin{aligned}
\int_0^1 x^{\frac{s}{2}-1}\Psi(x)\mathrm{d}x &= \int_0^1 x^{\frac{s}{2}-1}\left(\frac{1}{\sqrt{x}}\Psi\left(\frac{1}{x}\right)+\frac{1}{2\sqrt{x}}-\frac{1}{2}\right)\mathrm{d}x \\
&= \int_0^1 \left(x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)+\frac{1}{2}x^{\frac{s-3}{2}}-\frac{1}{2}x^{\frac{s}{2}-1}\right)\mathrm{d}x \\
&= \int_0^1 x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)\mathrm{d}x+\frac{1}{2}\int_0^1\left(x^{\frac{s-3}{2}}-x^{\frac{s}{2}-1}\right)\mathrm{d}x \\
&= \int_0^1 x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)\mathrm{d}x+\frac{1}{2}\left[\frac{2}{s-1}x^{\frac{s-1}{2}}-\frac{2}{s}x^{\frac{s}{2}}\right]_{x=0}^{x=1} \\
&= \int_0^1 x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)\mathrm{d}x+\frac{1}{2}\left[\frac{2}{s-1}(1)^{\frac{s-1}{2}}-\frac{2}{s}(1)^{\frac{s}{2}}-\frac{2}{s-1}(0)^{\frac{s-1}{2}}+\frac{2}{s}(0)^{\frac{s}{2}}\right] \\
&= \int_0^1 x^{\frac{s-3}{2}}\Psi\left(\frac{1}{x}\right)\mathrm{d}x+\frac{1}{s(s-1)}.
\end{aligned}$$

Furthermore, by changing the variable as $x \mapsto \frac{1}{u}$ on the right hand side, we obtain

$$\begin{aligned}
\int_0^1 x^{\frac{s}{2}-1}\Psi(x)\mathrm{d}x &= \int_\infty^1 \left(\frac{1}{u}\right)^{\frac{s-3}{2}}\Psi(u)(-u^{-2})\mathrm{d}u+\frac{1}{s(s-1)} \\
&= \int_1^\infty u^{\frac{-s+3}{2}}\Psi(u)u^{-2}\mathrm{d}u+\frac{1}{s(s-1)} \\
&= \int_1^\infty u^{\frac{-s-1}{2}}\Psi(u)\mathrm{d}u+\frac{1}{s(s-1)}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\Lambda(s) &= \int_0^1 x^{\frac{s}{2}-1}\Psi(x)\mathrm{d}x+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)\mathrm{d}x \\
&= \int_1^\infty x^{\frac{-s-1}{2}}\Psi(x)\mathrm{d}x+\frac{1}{s(s-1)}+\int_1^\infty x^{\frac{s}{2}-1}\Psi(x)\mathrm{d}x \\
&= \int_1^\infty \left(x^{\frac{-s-1}{2}}+x^{\frac{s}{2}-1}\right)\Psi(x)\mathrm{d}x+\frac{1}{s(s-1)} \\
&= \int_1^\infty \left(x^{\frac{-s+1}{2}}+x^{\frac{s}{2}}\right)x^{-1}\Psi(x)\mathrm{d}x+\frac{1}{s(s-1)}. \qquad (3.3.3)
\end{aligned}$$

Note that the integral on the right hand side is well-defined for any $s \in \mathbb{C}$, and so the right hand side is well defined for any $s \neq 0, 1$. This gives the analytic continuation of $\Lambda(s)$. Furthermore, it allows us to substitute $s \mapsto 1-s$, and therefore we can write Equation (3.3.3) as

$$\begin{aligned}\Lambda(1-s) &= \int_1^\infty \left(x^{\frac{-1+s+1}{2}} + x^{\frac{1-s}{2}}\right) x^{-1}\Psi(x)\mathrm{d}x + \frac{1}{(1-s)(1-s-1)} \\ &= \int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{-s+1}{2}}\right) x^{-1}\Psi(x)\mathrm{d}x + \frac{1}{(1-s)(s)} \\ &= \Lambda(s).\end{aligned}$$

Thus we get the desired result. □

Recall that the only possible poles on the right hand side of Equation (3.3.3) are at $s = 0$ and 1. Together with the fact that $\frac{1}{\Gamma(s)}$ is entire (as stated in Corollary 3.1.7), we observe that

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \left(\int_1^\infty \left(x^{\frac{s}{2}} + x^{\frac{-s+1}{2}}\right) x^{-1}\Psi(x)\mathrm{d}x + \frac{1}{(1-s)(s)}\right).$$

is meromorphic on $\mathbb{C}$, except possibly at $s = 0, 1$. We now claim that the pole at $s = 0$ is removable. This can be shown by the following expression:

$$\lim_{s\to 0} \frac{1}{s} \cdot \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} = \lim_{s\to 0} \frac{1}{2\cdot\frac{s}{2}} \cdot \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} = \lim_{s\to 0} \frac{\pi^{\frac{s}{2}}}{2\Gamma\left(\frac{s}{2}+1\right)} = \frac{1}{2\Gamma(1)} = \frac{1}{2}.$$

The second equality follows from Proposition 3.1.2. Hence, $\zeta(s)$ has an analytic continuation to $\mathbb{C}$ with a simple pole at $s = 1$. To end this chapter, we state the functional equation of the Riemann zeta function below.

Theorem 3.3.3 *For any complex number $s \neq 1$, the following function is well-defined:*

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s)\zeta(1-s).$$

Proof. This follows from Theorem 3.1.6, Proposition 3.1.8, and Proposition 3.3.2.

First, in Proposition 3.1.8, we apply a change of variable $s \mapsto \frac{s}{2}$ to get

$$\Gamma(s) = \frac{2^{s-1}}{\sqrt{\pi}} \Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s}{2} + \frac{1}{2}\right).$$

Therefore, we have

$$\Gamma\left(\frac{s}{2}\right) \Gamma\left(\frac{s+1}{2}\right) = \frac{\sqrt{\pi}}{2^{s-1}} \Gamma(s). \tag{3.3.4}$$

Also, substituting $s \mapsto \frac{s+1}{2}$ in Theorem 3.1.6, we have

$$\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(1 - \frac{s+1}{2}\right) = \frac{\pi}{\sin\left(\frac{\pi s}{2} + \frac{\pi}{2}\right)}. \tag{3.3.5}$$

Since $\frac{1}{\Gamma(s)}$ is entire, we can divide Equation (3.3.4) by Equation (3.3.5) to get

$$\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)} = \frac{2}{2^s\sqrt{\pi}}\cos\left(\frac{\pi s}{2}\right)\Gamma(s). \tag{3.3.6}$$

Lastly we manipulate the functional equation of Proposition 3.3.2 to have the left hand side of Equation (3.3.6), that is, to divide both sides of

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{s-1}{2}}\Gamma\left(\frac{s-1}{2}\right)\zeta(s-1)$$

by $\Gamma\left(\frac{s-1}{2}\right)$ and rearrange to get

$$\zeta(1-s) = \pi^{\frac{1}{2}-s}\frac{\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{1-s}{2}\right)}\zeta(s).$$

By substituting in Equation (3.3.6), we obtain our desired result. $\square$

Remark 3.3.4 *The* ***Dirichlet eta function*** $\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$ *for complex s where* $\Re(s) > 0$*, is also used to prove the analytical continuation of the Riemann zeta function to the critical strip where* $0 < Re(s) < 1$*. It is not explained in this note, but refer to the sources focusing on the Riemann Hypothesis, such as [18] and [***Kim_Riemann_Hypothesis***].*

Proposition 3.3.5 $\zeta(s)$ *has a simple pole at* $s = 1$*.*

Proof. An easy way to see this is to consider $\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s)$. The terms $2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\zeta(1-s)$ are well defined when $s = 1$, but $\Gamma(1-s)$ has a simple pole at $s = 1$. $\square$

Proposition 3.3.6 (The Zeta function's Trivial Zeroes) *The zeroes of* $\zeta(s)$ *in* $\Re(s) < 0$ *are where* s *is a negative even integer, that is,*

$$\zeta(-2k) = 0 \textit{ for } k \in \mathbb{N}.$$

Proof. Let s be a complex number and $\Re(s) < 0$. So the $\zeta(s)$ representation we use is

$$\zeta(s) = 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).$$

On the right hand side, $2^s\pi^{s-1}\zeta(1-s) \neq 0$, so we only have to look at $\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)$. However, $\Gamma(1-s)$ has simple poles for $1 - s = 0, -1, -2, \cdots$, so when $s = 1, 2, 3, \cdots$. Therefore when $\Re(s) < 0, \Gamma(1-s) \neq 0$ and is well-defined.Therefore when $\Re(s) < 0$, $\zeta(s)$ has zeroes when $\sin\left(\frac{\pi s}{2}\right) = 0$, which when $s = -2, -4, -6, \ldots$. $\square$

CHAPTER 4
BERNOULLI NUMBERS AND BERNOULLI POLYNOMIALS

As seen in Theorem 2.0.5, Bernoulli numbers are deeply connected to the special values of the Riemann zeta function. In this chapter, we study some important properties of Bernoulli numbers as well as the Bernoulli polynomials.

4.1 Bernoulli Numbers

Bernoulli numbers first appeared in the Jacob Bernoulli's book, *Ars Conjectandi*, where he studied the sums of the k-th powers of n integers, $S_k(n) = \sum_{i=1}^{n} i^k$. In this section, we will introduce how Bernoulli encountered such numbers and study their useful properties. The exposition of this section is borrowed from [2] and [22].

Let us look at the first few formulas for sums of integer powers.

$$
\begin{aligned}
S_0(n) &= n \\
S_1(n) &= \frac{n(n+1)}{2} = \frac{1}{2}n^2 + \frac{1}{2}n \\
S_2(n) &= \frac{n(n+1)(2n+1)}{6} = \frac{1}{3}n^3 + \frac{1}{2}n^2 + \frac{1}{6}n \\
S_3(n) &= \frac{n^2(n-1)^2}{4} = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 \\
S_4(n) &= \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30} = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n \\
S_5(n) &= \frac{n^2(2n^2-2n-1)(n-1)^2}{12} = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 \\
&\ \ \vdots
\end{aligned}
$$

Thus a pattern emerges,

$$
S_k(n) = \frac{1}{k+1}n^{k+1} + \frac{1}{2}n^k + \frac{k}{12}n^{k-1} + 0 \cdot n^{k-2} + \cdots , \tag{4.1.1}
$$

which we now write as

$$
\begin{aligned}
S_k(n) = &\frac{1}{k+1} B_0 n^{k+1} + B_1 n^k + \frac{k}{2} B_2 n^{k-1} + \frac{k(k-1)(k-2)}{2\cdot 3\cdot 4} B_4 n^{k-3} \\
&+ \frac{k(k-1)(k-2)(k-3)(k-4)}{2\cdot 3\cdot 4\cdot 5\cdot 6} B_6 n^{k-5} \\
&+ \frac{k(k-1)(k-2)(k-3)(k-4)(k-5)(k-6)}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8} B_8 n^{k-7} + \cdots ,
\end{aligned}
$$

where

$$B_0 = 1,\ B_1 = \frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30},\ B_5 = 0, B_6 = \frac{1}{42}, \ldots .$$

Such B_k's are called the k-th Bernoulli numbers. Indeed, these are called the "classical" Bernoulli numbers nowadays. Recent study frequently adopts so-called the "modern" Bernoulli numbers, which only differs from the classical definition at $k = 1$. We will use the modern definition for our later computation purposes.

Many define Bernoulli numbers using its generating function or recursive function. Here, we adopt the recursive definition, but later show that these are equivalent.

Definition 4.1.1 *Let $B_0 = 0$. For $k \geq 1$, the (modern) k-th **Bernoulli number** B_k is defined recursively as*

$$\sum_{k=0}^{n} \binom{n+1}{k} B_k = 0.$$

For example, the first six Bernoulli numbers are,

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_3 = 0,\ B_4 = \frac{-1}{30},\ B_5 = 0, B_6 = \frac{1}{42}, \ldots .$$

In what follows, we show that the Bernoulli numbers can be equivalently defined by its generating function.

Proposition 4.1.2 *The generating function below gives the k-th Bernoulli number B_k as a coefficient of $\frac{t^k}{k!}$.*

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}$$

Proof. From Definition 4.1.1, if $k \geq 1$, we see

$$\left(\sum_{j=0}^{k-1} \binom{k}{j} B_j \right) + B_k = 0 + B_k = B_k.$$

Note that the left hand side in the above expression can be also written as

$$\left(\sum_{j=0}^{k-1}\binom{k}{j}B_j\right)+\binom{k}{k}B_k=\sum_{j=0}^{k}\binom{k}{j}B_j.$$

Therefore we can express the k-th Bernoulli Number as

$$B_k=\sum_{j=0}^{k}\binom{k}{j}B_j. \tag{4.1.2}$$

On the other hand, let $t\neq 0$ and $g(t)=\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}$. Then applying Equation (4.1.2), we observe

$$\begin{aligned}
g(t)e^t &= \sum_{k=0}^{\infty}B_k\frac{t^k}{k!}\sum_{k=0}^{\infty}\frac{t^k}{k!}\\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}B_j\right)\frac{t^k}{k!}\\
&= \left(\binom{0}{0}B_0\right)\frac{t^0}{0!}+\left(\binom{1}{0}B_0+\binom{1}{1}B_1\right)\frac{t^1}{1!}\\
&\quad+\left(\binom{2}{0}B_0+\binom{2}{1}B_1+\binom{2}{2}B_2\right)\frac{t^2}{2!}+\sum_{k=3}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}B_j\right)\frac{t^k}{k!}\\
&= B_0+(B_0+B_1)t+(B_0+2\,B_1+B_2)\frac{t^2}{2!}+\sum_{k=3}^{\infty}B_k\frac{t^k}{k!}\\
&= 1+(1+B_1)t+(1-1+B_2)\frac{t^2}{2!}+\sum_{k=3}^{\infty}B_k\frac{t^k}{k!}\\
&= 1+t+B_1t+B_2\frac{t^2}{2!}+\sum_{k=3}^{\infty}B_k\frac{t^k}{k!}.
\end{aligned}$$

Since $1+B_1t=\frac{B_0t^0}{0!}+\frac{B_1t^1}{1!}$, we have

$$g(t)e^t=t+\sum_{k=0}^{\infty}B_k\frac{t^k}{k!}=t+g(t).$$

Thus,

$$g(t)e^t-g(t)=t,$$

or equivalently,

$$\frac{t}{e^t - 1} = g(t) = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

□

4.2 Bernoulli Polynomials

We now define Bernoulli polynomials. Since there is one accepted representation of Bernoulli polynomials, we will define them by their generating function and note important properties that relate to the Bernoulli numbers.

Definition 4.2.1 *The k-th **Bernoulli polynomials** $B_k(x)$ are defined as*

$$\sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \frac{te^{xt}}{e^t - 1}.$$

In this section, we prove some properties connecting Bernoulli polynomials with Bernoulli numbers. These properties can also be found in [22].

1. (Spawning Bernoulli Polynomials from Bernoulli Numbers)

$$B_k(x) = \sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j} \qquad \text{for } k \geq 0$$

Proof. Consider the following,

$$\sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} = \frac{te^{xt}}{e^t - 1} = \frac{t}{e^t - 1} \cdot e^{xt} = \left(\sum_{k=0}^{\infty} B_k \frac{t^k}{k!} \right) \left(\sum_{k=0}^{\infty} \frac{x^k t^k}{k!} \right)$$
$$= \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} \binom{k}{j} B_j x^{k-j} \right) \frac{t^k}{k!}.$$

The desired result follows by comparing coefficients. □

Using the previous property, we can easily compute the first few Bernoulli polynomials as follows:

$$B_0(x) = 1, B_1(x) = x - \frac{1}{2}, \ B_2(x) = x^2 - x + \frac{1}{6},$$
$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x, \ B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}, \ \ldots.$$

2. $B_k(x+1) - B_k(x) = kx^{k-1}$ for $k \geq 0$
   Proof. We express te^{xt} in two different ways. Indeed, we have

$$te^{xt} = t\sum_{k=0}^{\infty} \frac{x^k t^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k t^{k+1}}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1} t^k}{(k-1)!}.$$

   On the other hand, te^{xt} can also be written as

$$\begin{aligned} te^{xt} &= \frac{te^{xt}}{e^t - 1}(e^t - 1) = \frac{te^{t(x+1)}}{e^t - 1} - \frac{te^{xt}}{e^t - 1} \\ &= \sum_{k=0}^{\infty} B_k(x+1)\frac{t^k}{k!} - \sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty} \left(B_k(x+1) - B_k(x)\right)\frac{t^k}{k!}. \end{aligned}$$

   Comparing the two expressions yields

$$\sum_{k=0}^{\infty} \left(B_k(x+1) - B_k(x)\right)\frac{t^k}{k!} = \sum_{k=1}^{\infty} \frac{x^{k-1}t^k}{(k-1)!},$$

   and therefore
$$\frac{B_k(x+1) - B_k(x)}{k!} = \frac{x^{k-1}}{(k-1)!}.$$

   Multiplying both sides of the equation by $k!$, we obtain

$$B_k(x+1) - B_k(x) = \frac{k!\, x^{k-1}}{(k-1)!} = kx^{k-1}.$$

$\square$

3. $B_k(1-x) = (-1)^k B_k(x)$ for $k \geq 0$

Proof. The generating function for the Bernoulli numbers gives

$$\begin{aligned}
\sum_{k=0}^{\infty} B_k(1-x)\frac{t^k}{k!} = \frac{te^{(1-x)t}}{e^t-1} &= \frac{te^t}{e^t-1}e^{-xt} \\
&= \frac{e^t t}{e^t(1-e^{-t})}\frac{-1}{-1}e^{-xt} = \frac{-t}{e^{-t}-1}e^{-xt} \\
&= \left(\sum_{k=0}^{\infty} B_k \frac{(-t)^k}{k!}\right)\left(\sum_{k=0}^{\infty} \frac{(-x)^k t^k}{k!}\right) \\
&= \sum_{k=0}^{\infty}\left(\sum_{j=0}^{k}\binom{k}{j}(-1)^j B_j(-1)^{k-j}x^{k-j}\right)\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty}(-1)^k\left(\sum_{j=0}^{k}\binom{k}{j}B_j^{k-j}\right)\frac{t^k}{k!} \\
&= \sum_{k=0}^{\infty}(-1)^k B_k(x)\frac{t^k}{k!}.
\end{aligned}$$

Note that the Bernoulli polynomial Property 1 was applied in the second and fourth lines in the above equation. We get our desired result by comparing coefficients. □

4. $B_k'(x) = kB_{k-1}(x)$ for $k \geq 1$ and $B_0'(x) = 0$

Proof. Since $B_0(x) = 1$, it is clear that $B_0'(x) = 0$. Now let $k \geq 1$ and take the partial derivative of the generating function with respect to x,

$$\frac{\partial}{\partial x}\left(\frac{te^{xt}}{e^t-1}\right) = \frac{\partial}{\partial x}\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!} = \sum_{k=0}^{\infty}\frac{\partial}{\partial x}\left(B_k(x)\frac{t^k}{k!}\right) = \sum_{k=0}^{\infty} B_k'(x)\frac{t^k}{k!}.$$

On the other hand,

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{te^{xt}}{e^t-1}\right) = \frac{t^2e^{xt}}{e^t-1} &= t\left(\frac{te^{xt}}{e^t-1}\right) \\
&= t\left(\sum_{k=0}^{\infty} B_k(x)\frac{t^k}{k!}\right) = \left(\sum_{k=0}^{\infty} B_k(x)\frac{t^{k+1}}{k!}\right) \\
&= \left(\sum_{k=1}^{\infty} B_{k-1}(x)\frac{t^k}{(k-1)!}\right).
\end{aligned}$$

Thus,

$$\sum_{k=0}^{\infty} B_k'(x)\frac{t^k}{k!} = \sum_{k=1}^{\infty} B_{k-1}(x)\frac{t^k}{(k-1)!}.$$

Therefore by comparing coefficients of the t^k-th term, we have

$$\frac{B_k'(x)}{k!} = \frac{B_{k-1}(x)}{(k-1)!},$$

or equivalently,

$$B_k'(x) = \frac{k!\,B_{k-1}(x)}{(k-1)!} = kB_{k-1}(x).$$

□

5. $B_k(0) = B_k$ and $B_k(1) = (-1)^k B_k$ for $k \geq 1$

   Proof. The first part follows from substituting $x = 0$ into Definition 4.2.1 and Proposition 4.1.2. More precisely,

$$\sum_{k=0}^{\infty} B_k(0)\frac{t^k}{k!} = \frac{te^{(0)t}}{e^t - 1} = \frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

   For the second part, we again use Proposition 4.1.2,

$$\sum_{k=0}^{\infty} B_k(1)\frac{t^k}{k!} = \frac{te^{(1)t}}{e^t - 1} = \frac{(-e^{-t})te^t}{(-e^{-t})(e^t - 1)} = \frac{-t}{e^{-t} - 1} = \sum_{k=0}^{\infty}(-1)^k B_k \frac{t^k}{k!}.$$

   Thus we can compare coefficients to get the desired result. □

6. $\int_0^1 B_k(x)dx = 0$ for $k \geq 1$

   Proof. We can evaluate the integral by using Property 4 as follows:

$$\begin{aligned}\int_0^1 B_k(x)\mathrm{d}x = \int_0^1 \frac{1}{k+1}B_{k+1}'(x)\mathrm{d}x = \frac{1}{k+1}\int_0^1 B_{k+1}'(x)\mathrm{d}x \\ = \frac{1}{k+1}\left(B_{k+1}(1) - B_{k+1}(0)\right) \\ = \frac{1}{k+1}\left((-1)^{k+1}B_{k+1} - B_{k+1}\right).\end{aligned}$$

The last equality follows from Property 5. Finally, we note that

$$(-1)^{k+1}B_{k+1} - B_{k+1} = \begin{cases} 0 - 0 = 0 & \text{if } k \text{ is even} \\ B_{k+1} - B_{k+1} = 0 & \text{if } k \text{ is odd,} \end{cases}$$

which completes the proof. $\square$

CHAPTER 5
SPECIAL VALUES OF RIEMANN ZETA FUNCTION

One of the areas studied extensively in the seventeenth century was to understand infinite series. In 1644, Pietro Mengoli introduced the Basel problem, which is to evaluate $\sum_{n=1}^{\infty} \frac{1}{n^2}$, now known as the value of $\zeta(2)$. However, it was ignored until Johann Bernoulli published it in 1689. It remained unsolved until 1734, when Euler showed that $\zeta(2) = \frac{\pi^2}{6}$. Furthermore, he found a way to evaluate $\zeta(2k)$ for any positive integer k. In this section, we will explain his method and a newly published method for evaluating $\zeta(2k)$.

5.1 Connecting to the Cotangent Function

In this section, we follow one of Euler's proofs described above. This can also be found in, for example, [20], [9], and [15].

Note that $\frac{\sin(\pi s)}{\pi s}$ has infinitely many zeroes when $s \in \{\ldots, -2, -1, 1, 2, \ldots\}$. In order to describe the function as a product of its linear factors, we apply the following theorem proved by the nineteenth century mathematician Karl Weierstrass. The theorem is listed below, and more information can be found on page 71 of [9].

Theorem 5.1.1 (Weierstrass Factor Theorem) *Let $f(s)$ be an entire function with a zero at $s = 0$ of order $b \geq 0$ and non-zero zeroes $s = c_1, c_2, c_3, \ldots$. There exists a sequence of numbers $a_1, a_2, a_3, \ldots$ and an entire function $g(s)$, $f(s)$ such that*

$$f(s) = s^b e^{g(s)} \prod_{n=1}^{\infty} E_{a_n}\left(\frac{s}{c_n}\right).$$

Where for an a in the sequence $a_1, a_2, a_3, \ldots$,

$$E_a(s) = \begin{cases} (1-s) & \text{if } a = 0 \\ (1-s)e^{\sum_{a=1}^{a} \frac{s^a}{a}} & \text{otherwise.} \end{cases}$$

Therefore, we can write $\sin(s)$ as

$$\sin(\pi s) = \pi s \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} e^{\frac{s}{n}} \left(1 - \frac{s}{n}\right).$$

and thus,

$$\frac{\sin(\pi s)}{\pi s} = \prod_{n=1}^{\infty} \left(1 - \frac{s^2}{n^2}\right). \tag{5.1.1}$$

Multiplying them out and rearranging terms according to the powers of s, we get that the coefficient of s^{2k} is the sum of reciprocals of the product of k squared natural numbers multiplied by $(-1)^k$, so

$$\frac{\sin(\pi s)}{\pi s} = \sum_{k=1}^{\infty}(-1)^k \left(\sum_{n_1,\ldots,n_k \in \mathbb{N}} \frac{1}{n_1^2 \cdots n_k^2} \right) s^{2k},$$

or more explicitly for our purposes,

$$\frac{\sin(\pi s)}{\pi s} = 1 - \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \right) s^2 \tag{5.1.2}$$

$$+ \left(\frac{1}{1^2 2^2} + \frac{1}{1^2 3^2} + \frac{1}{2^3 3^2} + \cdots \right) s^4 - \left(\frac{1}{1^2 2^2 3^2} + \cdots \right) s^6 + \cdots . \tag{5.1.3}$$

Therefore, we can see that the coefficient of s^2 is $\zeta(2)$. We now recall the Taylor expansion of the sine function,

$$\sin(s) = s - \frac{s^3}{3!} + \frac{s^5}{5!} - \frac{s^7}{7!} + \cdots .$$

Now by letting $s \mapsto \pi s$ and dividing by πs, we have

$$\begin{aligned}\frac{\sin(\pi s)}{\pi s} &= 1 - \frac{(\pi s)^2}{3!} + \frac{(\pi s)^4}{5!} - \frac{(\pi s)^6}{7!} + \cdots \\ &= 1 - \frac{\pi^2}{3!} s^2 + \frac{\pi^4}{5!} s^4 - \frac{\pi^6}{7!} s^6 + \cdots .\end{aligned}$$

Since the Taylor expansion is unique, we can compare the coefficients of the Taylor expansion and Equation (5.1.2) to find the value of $\zeta(2)$. More precisely, we observe that $-\zeta(2)s^2 = \frac{-\pi^2}{3!}s^2$, which allows us to conclude

$$\zeta(2) = \frac{\pi^2}{6}.$$

Remark 5.1.2 *The Weierstrass factor theorem was not known during the time of Euler, yet Euler assumed this property was true, as stated by in [21]. Indeed, it was about a hundred years later when Weierstrass rigorously proved this theorem. What Euler knew was that if $p(x)$ is an n degree polynomial such that it has n non zero roots, $c_1, \ldots, c_n$, and $p(0) = 1$, then $p(x)$ can be written as*

$$p(x) = \prod_{i=1}^{n} \left(1 - \frac{x}{c_i} \right).$$

Then Euler assumed that this holds for an infinite degree polynomial and wrote

$$\begin{aligned}\frac{\sin(\pi s)}{\pi s} &= \left(1 - \frac{s}{1}\right)\left(1 + \frac{s}{1}\right)\left(1 - \frac{s}{2}\right)\left(1 + \frac{s}{2}\right)\left(1 - \frac{s}{3}\right)\left(1 + \frac{s}{3}\right)\cdots \\ &= \left(1 - \frac{s^2}{1^2}\right)\left(1 - \frac{s^2}{2^2}\right)\left(1 - \frac{s^2}{3^2}\right)\cdots .\end{aligned}$$

Euler's method for evaluating $\zeta(2)$ demonstrates how to find some values of the Riemann zeta function using the properties of trigonometric functions. Now we generalize his approach to prove Theorem 2.0.5.

This approach consists of three parts. First, we connect $\zeta(2k)$ with the cotangent function. Secondly, we connect the cotangent function with Bernoulli numbers. Lastly, we show the relationship between Bernoulli numbers and the Riemann zeta function, using the cotangent relationships as bridges. This proof is a more detailed version of Chapter 9.6 of [15].

The first step is to express special values of $\zeta(2k)$ in terms of the cotangent function.

Proposition 5.1.3 *For natural number k and $s \in (0,1)$,*

$$\pi s \cdot \cot(\pi s) = 1 - 2\sum_{k=1}^{\infty} \zeta(2k) s^{2k}.$$

Proof. Let s be a real number greater than 0 and less than 1. By taking the logarithm of both sides of Equation (5.1.1), we have

$$\log\left(\frac{\sin(\pi s)}{\pi s}\right) = \log\left(\prod_{n=1}^{\infty}\left(1 - \frac{s^2}{n^2}\right)\right) = \sum_{n=1}^{\infty} \log\left(1 - \frac{s^2}{n^2}\right),$$

and therefore

$$\log\left(\sin(\pi s)\right) = \log(\pi s) + \sum_{n=1}^{\infty} \log\left(1 - \frac{s^2}{n^2}\right).$$

Taking the derivative of the above equation with respect to s gives us the cotangent function because $\frac{\mathrm{d}}{\mathrm{d}s}\log\left(\sin(\pi s)\right) = \frac{\cos(\pi s)}{\sin(\pi s)}\pi$. On the other hand,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\log(\pi s) + \sum_{n=1}^{\infty} \log\left(1 - \frac{s^2}{n^2}\right)\right) = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{1}{1 - \dfrac{s^2}{n^2}} \cdot \left(\frac{-2s}{n^2}\right).$$

Therefore, we obtain

$$\frac{\cos(\pi s)}{\sin(\pi s)}\pi = \frac{1}{s} + \sum_{n=1}^{\infty} \frac{1}{1 - \dfrac{s^2}{n^2}} \cdot \left(\frac{-2s}{n^2}\right)$$

Multiply both sides by s to obtain

$$\pi s \cdot \cot(\pi s) = 1 + \sum_{n=1}^{\infty} \frac{1}{1 - \dfrac{s^2}{n^2}} \cdot \left(\frac{-2s^2}{n^2}\right), \tag{5.1.4}$$

which gives us a nice series expression for $\pi s \cdot \cot(\pi s)$. Since $\left|\frac{s^2}{n^2}\right| < 1$ with our assumption, we may write $\frac{1}{1 - \frac{s^2}{n^2}}$ as a geometric series,

$$\sum_{k=0}^{\infty}\left(\frac{s^2}{n^2}\right)^k = \frac{1}{1 - \frac{s^2}{n^2}}.$$

Substituting it into Equation (5.1.4) gives us

$$\begin{aligned}\pi s\cdot\cot(\pi s) &= 1+\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{s^2}{n^2}\right)^k\left(\frac{-2s^2}{n^2}\right)\\ &= 1-2\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\left(\frac{s^2}{n^2}\right)^{k+1}\\ &= 1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\frac{s^2}{n^2}\right)^{k}\\ &= 1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{1}{n^{2k}}\cdot s^{2k}.\end{aligned}$$

In the above equation, we can justify changing the order of summations by knowing that for $s\in(0,1)$, $\sum_{n=1}^{\infty}\frac{1}{n^{2k}}$ and $\sum_{k=1}^{\infty}s^{2k}$ are absolutely convergent series. Therefore, we can write

$$\pi s\cdot\cot(\pi s)=1-2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{n^{2k}}\cdot s^{2k}=1-2\sum_{k=1}^{\infty}\zeta(2k)\cdot s^{2k}.$$

This completes the proof.

For our second step, we prove relationship between Bernoulli numbers and the cotangent function. This proof comes from the Appendix of [2] and from [10].

Proposition 5.1.4 *For $s\in(0,1)$, the following is true:*

$$\pi s\cdot\cot(\pi s)=i\pi s+\sum_{k=0}^{\infty}B_k\frac{(2i\pi s)^k}{k!}.$$

Proof. First we express $\pi s\cdot\cot(\pi s)$ using the identity $\cot(\pi s)=\cos(\pi s)/\sin(\pi s)$ and the Euler formula for complex exponentials. Indeed, we have

$$\begin{aligned}\pi s\cdot\cot(\pi s) &= \pi s\cdot\frac{\cos(\pi s)}{\sin(\pi s)}=\pi s\cdot\frac{\frac{e^{i\pi s}+e^{-i\pi s}}{2}}{\frac{e^{i\pi s}-e^{i\pi s}}{2i}}\\ &= i\pi s\cdot\frac{e^{i\pi s}+e^{-i\pi s}}{e^{i\pi s}-e^{i\pi s}}=i\pi s\cdot\frac{e^{2i\pi s}+1}{e^{2i\pi s}-1}\\ &= i\pi s\cdot\frac{e^{2i\pi s}-1+2}{e^{2i\pi s}-1}\\ &= i\pi s\cdot\left(1+\frac{2}{e^{2i\pi s}-1}\right)\\ &= i\pi s+\frac{2i\pi s}{e^{2i\pi s}-1}.\end{aligned}$$

Notice that the last term in the expression above is nothing but the Bernoulli number generating function in Proposition 4.1.2 with $t = 2i\pi s$. This substitution gives us

$$\pi s \cdot \cot(\pi s) = i\pi s + \sum_{k=0}^{\infty} B_k \frac{(2i\pi s)^k}{k!}.$$

Propositions 5.1.4 and 5.1.3 give the following relation,

$$i\pi s + \sum_{k=0}^{\infty} B_k \frac{(2i\pi s)^k}{k!} = \pi s \cdot \cot(\pi s) = 1 - 2\sum_{k=1}^{\infty} \zeta(2k) s^{2k}. \tag{5.1.5}$$

As for our final step, we now make a further observation on the left hand side of Equation 5.1.5, rewriting it as

$$\begin{aligned}
i\pi s + \sum_{k=0}^{\infty} B_k \frac{(2i\pi s)^k}{k!} &= i\pi s + \frac{B_0}{0!} + 2\frac{B_1}{1!}(i\pi s) + \sum_{k=2}^{\infty} B_k \frac{(2i\pi s)^k}{k!} \\
&= i\pi s + \frac{1}{1} + 2\frac{-1/2}{1}(i\pi s) - 2\sum_{k=2}^{\infty} \frac{-1}{2} B_k \frac{(2i\pi s)^k}{k!} \\
&= i\pi s + 1 - i\pi s - 2\sum_{k=2}^{\infty} \frac{-1}{2} B_k \frac{(2i\pi s)^k}{k!} \\
&= 1 - 2\sum_{k=2}^{\infty} \frac{-1}{2} B_k \frac{(2i\pi s)^k}{k!}.
\end{aligned}$$

Noting that $B_{2k+1} = 0$ for $k > 0$, we have

$$\begin{aligned}
1 - 2\sum_{k=1}^{\infty} \frac{-1}{2} B_{2k} \frac{(2i\pi s)^{2k}}{(2k)!} &= 1 - 2\sum_{k=1}^{\infty} \frac{-1}{2} B_{2k} \frac{(2\pi)^{2k} i^{2k} s^{2k}}{(2k)!} \\
&= 1 - 2\sum_{k=1}^{\infty} \frac{-1}{2} B_{2k} \frac{(2\pi)^{2k} (-1)^k s^{2k}}{(2k)!} \\
&= 1 - 2\sum_{k=1}^{\infty} B_{2k} \frac{(-1)^{k+1} (2\pi)^{2k} s^{2k}}{2(2k)!}.
\end{aligned}$$

Comparing coefficients of this expression with the right hand side of Equation (5.1.5) provides that

$$\zeta(2k) = B_{2k} \frac{(-1)^{k+1} (2\pi)^{2k}}{2(2k)!}.$$

This completes the proof of Theorem 2.0.5.

5.2 Alternative Proof

In May 2015, Óscar Ciaurri, Luis Navas, Francisco Ruiz and Juan Varona published a paper [11], providing a new method to evaluate $\zeta(2k)$ that only uses properties of Bernoulli polynomials and that of telescoping series. It is remarkable that this new method does not require knowledge of complex analysis. In this section, we study their proof thoroughly.

First let us define the integral $\mathrm{I}(k,m)$. For any integers $k \geq 0$ and $m \geq 1$,

$$\mathrm{I}(k,m) := \int_0^1 B_{2k}(t) \cdot \cos(m\pi t)\mathrm{d}t. \tag{5.2.1}$$

Notice that $\mathrm{I}(0,m) = 0$ because

$$\begin{aligned}\mathrm{I}(0,m) &= \int_0^1 B_0(t) \cdot \cos(m\pi t)\mathrm{d}t \\ &= \int_0^1 1 \cdot \cos(m\pi t)\mathrm{d}t \\ &= 0.\end{aligned}$$

We now find $\mathrm{I}(k,m)$ for $k \geq 1$ using integration by parts twice. Indeed,

$$\begin{aligned}\mathrm{I}(k,m) &= \left[B_{2k}(t) \cdot \frac{1}{m\pi}\sin(m\pi t)\right]_{t=0}^{t=1} - \int_0^1 \frac{1}{m\pi}\sin(m\pi t) \cdot 2k \cdot B_{2k-1}(t)\mathrm{d}t \qquad (5.2.2) \\ &= \frac{1}{m\pi}\left[B_{2k}(t) \cdot \sin(m\pi t)\right]_{t=0}^{t=1} - \frac{2k}{m\pi}\int_0^1 B_{2k-1}(t) \cdot \sin(m\pi t)\mathrm{d}t \\ &= -\frac{2k}{m\pi}\int_0^1 B_{2k-1}(t) \cdot \sin(m\pi t)\mathrm{d}t.\end{aligned}$$

Applying integration by parts in the integral on the right hand side, we observe

$$\begin{aligned}
\int_0^1 B_{2k-1}(t)\cdot\sin(m\pi t)\mathrm{d}t &= \left[B_{2k-1}(t)\cdot\Big(-\frac{\cos(m\pi t)}{m\pi}\Big)\right]_{t=0}^{t=1} \\
&\qquad - \int_0^1 \Big(-\frac{\cos(m\pi t)}{m\pi}\Big)\cdot\Big((2k-1)B_{2k-2}(t)\mathrm{d}t\Big) \\
&= \frac{-1}{m\pi}\left[B_{2k-1}(t)\cdot\cos(m\pi t)\right]_{t=0}^{t=1} + \frac{2k-1}{m\pi}\int_0^1 B_{2k-2}(t)\cdot\cos(m\pi t)\mathrm{d}t \\
&= \frac{-1}{m\pi}\left[B_{2k-1}(t)\cdot\cos(m\pi t)\right]_{t=0}^{t=1} + \frac{2k-1}{m\pi}\mathrm{I}(k-1,m).
\end{aligned}$$

Substituting the above equation back into Equation (5.2.2), we obtain

$$\begin{aligned}
\mathrm{I}(k,m) &= -\frac{2k}{m\pi}\left(\frac{-1}{m\pi}\left[B_{2k-1}(t)\cdot\cos(m\pi t)\right]_{t=0}^{t=1} + \frac{2k-1}{m\pi}\mathrm{I}(k-1,m)\right) \\
&= \frac{2k}{m^2\pi^2}\left[B_{2k-1}(t)\cdot\cos(m\pi t)\right]_{t=0}^{t=1} - \frac{2k(2k-1)}{m^2\pi^2}\,\mathrm{I}(k-1,m) \\
&= \frac{2k}{m^2\pi^2}\left(B_{2k-1}(1)\cdot\cos(m\pi) - B_{2k-1}(0)\cdot\cos(0)\right) - \frac{2k(2k-1)}{m^2\pi^2}\,\mathrm{I}(k-1,m) \\
&= \frac{2k}{m^2\pi^2}\left(B_{2k-1}(1)\cdot\cos(m\pi) - B_{2k-1}\right) - \frac{2k(2k-1)}{m^2\pi^2}\,\mathrm{I}(k-1,m).
\end{aligned}$$

Notice that, if $k = 1$, we have

$$\begin{aligned}
\mathrm{I}(1,m) &= \frac{2k}{m^2\pi^2}\left(B_1(1)\cdot\cos(m\pi) - B_1\right) - \frac{2}{m^2\pi^2}\,\mathrm{I}(0,m) \\
&= \frac{2k}{m^2\pi^2}\left(B_1(1)\cdot\cos(m\pi) - B_1\right) \\
&= \frac{2}{m^2\pi^2}\left[\frac{1}{2}\cdot\cos(m\pi) - \Big(\frac{-1}{2}\Big)\right] \\
&= \frac{2}{m^2\pi^2}\left[\frac{1}{2}\cdot\cos(m\pi) + \frac{1}{2}\right] \\
&= \frac{2}{m^2\pi^2}\left((-1)^m\frac{1}{2} + \frac{1}{2}\right).
\end{aligned}$$

Therefore, we get an explicit expression for $\mathrm{I}(1, m)$ as

$$\mathrm{I}(1, m) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \dfrac{2}{m^2\pi^2} & \text{if } m \text{ is even.} \end{cases}$$

From Bernoulli Polynomials' Property 5, we know that $B_{2k-1}(1) = (-1)^{2k-1}B_{2k-1}$. Since $B_{2k-1} = 0$ for $k \geq 2$, we have $B_{2k-1}(1) = 0$. Therefore we have the following recurrence relation, for $k \geq 2$,

$$\mathrm{I}(k, m) = \frac{2k(2k-1)}{m^2\pi^2}\,\mathrm{I}(k-1, m). \tag{5.2.3}$$

Together with the evaluation for $\mathrm{I}(1, m)$, we now have an explicit expression for $\mathrm{I}(k, m)$ for any $k \geq 1$ as

$$\mathrm{I}(k, m) = \begin{cases} 0 & \text{if } m \text{ is odd} \\ \dfrac{(-1)^{k-1}(2k)!}{m^{2k}\pi^{2k}} & \text{if } m \text{ is even.} \end{cases} \tag{5.2.4}$$

Next, we define $\mathrm{I}^*(k, m)$ by slightly modifying $\mathrm{I}(k, m)$. The importance of $\mathrm{I}^*(k, m)$ will come up later when we discuss the telescoping series. Define $B_k^*(t)$ as

$$B_k^*(t) := B_k(t) - B_k(0) = B_k(t) - B_k,$$

and let

$$\mathrm{I}^*(k, m) := \int_0^1 B_{2k}^*(t) \cdot \cos(m\pi t)\mathrm{d}t \tag{5.2.5}$$

for integers $k \geq 0$ and $m \geq 1$.

Notice that since $\int_0^1 \cos(m\pi t)\mathrm{d}t = 0$ for $m \geq 1$,

$$\begin{aligned} \mathrm{I}^*(k, m) &= \int_0^1 B_{2k}^*(t) \cdot \cos(m\pi t)\mathrm{d}t \\ &= \int_0^1 B_{2k}(t) \cdot \cos(m\pi t)\mathrm{d}t - B_{2k}\int_0^1 \cos(m\pi t)\mathrm{d}t \\ &= \mathrm{I}(k, m). \end{aligned}$$

Therefore although $\mathrm{I}(k, m)$ and $\mathrm{I}^*(k, m)$ have different representations, they evaluate equivalently. Now for $\mathrm{I}^*(k, m)$, fix some $k \geq 1$ and sum over positive integers m. Since $\mathrm{I}(k, m) = 0$

for odd m, we have

$$\begin{aligned}
\sum_{m=1}^{\infty} \mathrm{I}^*(k,m) &= \sum_{m=1}^{\infty} \mathrm{I}(k,m) \\
&= \mathrm{I}(k,1) + \mathrm{I}(k,2) + \mathrm{I}(k,3) + \mathrm{I}(k,4) + \mathrm{I}(k,5) + \cdots \\
&= \mathrm{I}(k,2) + \mathrm{I}(k,4) + \cdots \\
&= \sum_{m=1}^{\infty} \mathrm{I}(k,2m) \\
&= \sum_{m=1}^{\infty} \frac{(-1)^{k-1}(2k)!}{(2m)^{2k}\pi^{2k}} \\
&= \frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}} \sum_{m=1}^{\infty} \frac{1}{m^{2k}} \\
&= \frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}} \zeta(2k).
\end{aligned}$$

Therefore we have completed the first part of our proof with

$$\sum_{m=1}^{\infty} \int_0^1 B_{2k}^*(t) \cdot \cos(m\pi t)\mathrm{d}t = \sum_{m=1}^{\infty} \mathrm{I}^*(k,m) = \frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}} \zeta(2k) \tag{5.2.6}$$

To evaluate $\sum_{m=1}^{\infty} \int_0^1 B_{2k}^*(t) \cdot \cos(m\pi t)\mathrm{d}t$, we employ a trick of telescoping series. We will use the following trigonometric identity,

$$\cos(a) \cdot \sin(b) = \frac{1}{2}[\sin(a+b) - \sin(a-b)].$$

By taking $a = m\pi t$ and $b = \frac{\pi t}{2}$ with $t \in (0,1)$, we have

$$\cos(m\pi t) = \frac{\sin\left(\frac{2m+1}{2}\pi t\right) - \sin\left(\frac{2m-1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}. \tag{5.2.7}$$

Thus we can write

$$\begin{aligned}
\sum_{m=1}^{\infty} \mathrm{I}^*(k,m) &= \sum_{m=1}^{\infty} \int_0^1 B_{2k}^*(t) \cdot \cos(m\pi t)\mathrm{d}t \\
&= \lim_{N\to\infty} \sum_{m=1}^{N} \left[\int_0^1 B_{2k}^*(t) \frac{\sin\left(\frac{2m+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t - \int_0^1 B_{2k}^*(t) \frac{\sin\left(\frac{2m-1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t \right].
\end{aligned}$$

To show the telescoping behavior, consider $\mathrm{I}^*(k,a)$ and $\mathrm{I}^*(k,a+1)$. Then

$$\mathrm{I}^*(k,a)=\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2a+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t-\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2a-1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t,$$

and

$$\begin{aligned}\mathrm{I}^*(k,a+1)&=\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2(a+1)+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t-\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2(a+1)-1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t\\&=\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2a+3}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t-\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2a+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t.\end{aligned}$$

Adding $\mathrm{I}^*(k,a)$ with $\mathrm{I}^*(k,a+1)$, we have

$$\mathrm{I}^*(k,a)+\mathrm{I}^*(k,a+1)=-\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2a-1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t+\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2a+3}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t,$$

as the first term of $\mathrm{I}^*(k,a)$ is canceled by the second term of $\mathrm{I}^*(k,a+1)$. Thus

$$\begin{aligned}&\lim_{N\to\infty}\sum_{m=1}^{N}\left[\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2m+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t-\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2m-1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t\right]\\&=\lim_{N\to\infty}\left[-\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{\pi t}{2}\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t+\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2N+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t\right]\\&=-\frac{1}{2}\int_0^1 B_{2k}^*(t)\mathrm{d}t+\left[\lim_{N\to\infty}\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2N+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t\right].\end{aligned}$$

The first term can be evaluated as

$$-\frac{1}{2}\int_0^1 B_{2k}^*(t)\mathrm{d}t=-\frac{1}{2}\int_0^1(B_{2k}(t)-B_{2k})\mathrm{d}t=-\frac{1}{2}(-B_{2k})=\frac{B_{2k}}{2}.$$

This follows from Bernoulli polynomial Property 6 that shows $\int_0^1 B_{2k}(t)=0$. Now we want to show that

$$\lim_{N\to\infty}\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2N+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t=0.$$

Although $\frac{1}{\sin\left(\frac{\pi t}{2}\right)}$ is undefined for $t=0$, we consider the function

$$f(t)=\frac{B_{2k}^*(t)}{2\sin\left(\frac{\pi t}{2}\right)}$$

defined for $t\in(0,1]$. If we can find $\lim\limits_{\epsilon\to 0^+}f(\epsilon)$, then we can give an extension of $f(t)$ by continuity to $t = 0$. Since $B_{2k}^*(0) = B_{2k}(0) - B_{2k} = 0$ by Bernoulli polynomial Property 5, we may apply L'Hôpital's rule for $f(t)$ as

$$\begin{aligned}\lim_{\epsilon\to 0^+} f(0) &= \lim_{\epsilon\to 0^+} \frac{B_{2k}^*(\epsilon)}{2\sin\left(\frac{\pi\epsilon}{2}\right)} \\ &= \lim_{\epsilon\to 0^+} \frac{B_{2k}(\epsilon) - B_{2k}}{2\sin\left(\frac{\pi\epsilon}{2}\right)} \\ &= \lim_{\epsilon\to 0^+} \frac{2kB_{2k-1}(\epsilon)}{\pi\cos\left(\frac{\pi}{2}\epsilon\right)} \\ &= \frac{0}{\pi\cdot 1} = 0.\end{aligned}$$

So we now can extend $f(t)$ to $t \in [0, 1]$ as

$$f(t) = \begin{cases} 0 & t = 0 \\ \frac{B_{2k}^*(t)}{2\sin\left(\frac{\pi t}{2}\right)} & \text{otherwise.}\end{cases}$$

Let $R = \frac{(2N+1)\pi}{2}$. We now evaluate $\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2N+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t = \int_0^1 f(t)\cdot\sin(Rt)\mathrm{d}t$ using integration by parts.

$$\begin{aligned}\int_0^1 f(t)\cdot\sin(Rt)\mathrm{d}t &= \left[f(t)\left(-\frac{\cos(Rt)}{R}\right)\right]_{t=0}^{t=1} - \int_0^1 f'(t)\Big(-\frac{\cos(Rt)}{R}\Big)\mathrm{d}t \\ &= \left[f(t)\left(-\frac{\cos(Rt)}{R}\right)\right]_{t=0}^{t=1} + \int_0^1 f'(t)\Big(\frac{\cos(Rt)}{R}\Big)\mathrm{d}t \\ &= -f(1)\frac{\cos(R)}{R} + f(0)\frac{\cos(0)}{R} + \int_0^1 f'(t)\frac{\cos(Rt)}{R}\mathrm{d}t \\ &= -f(1)\frac{\cos(R)}{R} + \frac{1}{R}f(0) + \int_0^1 f'(t)\frac{\cos(Rt)}{R}\mathrm{d}t.\end{aligned}$$

From our original substitution, $R = \frac{(2N+1)\pi}{2}$, we observe each term in the sum approaches 0 as $N \to \infty$ because its numerator is bounded while its denominator tends to infinity. Thus,

$$\lim_{N\to\infty}\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2N+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t = \lim_{R\to\infty}\int_0^1 f(t)\sin(Rt)\mathrm{d}t = 0.$$

Hence we see that, by Equation (5.2.6),

$$\frac{(-1)^{k-1}(2k)!}{(2\pi)^{2k}}\zeta(2k) = -\frac{1}{2}\int_0^1 B_{2k}^*(t)\mathrm{d}t + \left[\lim_{N\to\infty}\int_0^1 B_{2k}^*(t)\frac{\sin\left(\frac{2N+1}{2}\pi t\right)}{2\sin\left(\frac{\pi t}{2}\right)}\mathrm{d}t\right] = \frac{B_{2k}}{2},$$

and therefore,

$$\zeta(2k) = \frac{(-1)^{k-1} 2^{2k} \pi^{2k}}{2(2k)!} B_{2k}, \qquad k = 1, 2, 3, \ldots.$$

CHAPTER 6
DIRICHLET L-FUNCTION

Following Chapter 7 of [17], Chapter 4 of [2], and [22], we study the basic properties of Dirichlet L-functions. We start with defining the Dirichlet character χ.

Definition 6.0.1 *A **Dirichlet character** χ modulo M is a multiplicative function that maps $(\mathbb{Z}/M\mathbb{Z})^\times$ to the complex plane with absolute value 1. Such a character can be extended as a function on all integers as the following:*

$$\chi(a+bM) = \begin{cases} \chi(a) & \textit{if } \gcd(a,M)=1 \\ 0 & \textit{otherwise.} \end{cases}$$

Therefore, a Dirichlet character extended to all integers has the following properties.

Proposition 6.0.2 *Let χ be a Dirichlet character modulo M that is extended to all integers. Then*

1. *χ is completely multiplicative, that is $\chi(ab) = \chi(a)\chi(b)$ for any integers a and b.*
2. *χ has a period of M.*
3. *If a and M are coprime, then $\chi(a) \neq 0$.*

A character is called, the **trivial character** χ_0 modulo M, if

$$\chi_0(a) = \begin{cases} 0 & \text{if } \gcd(a,M) > 1 \\ 1 & \text{if } \gcd(a,M) = 1. \end{cases}$$

In particular, the trivial character modulo 1 is called the **principal character** and denoted as $\chi = \mathbb{1}$.

Let $\phi(M)$ be the Euler totient function, that is

$$\phi(M) = \Big|\{n \in [1,M] : \gcd(n,M) = 1\}\Big|.$$

A well known property of the totient function is that if c and M are coprime, then $c^{\phi(M)} \equiv 1 \mod (M)$. Furthermore, we also see the following Lemma associating the Dirichlet character with the Euler totient function.

Lemma 6.0.3 *Let χ be a Dirichlet character modulo M. Then χ satisfies the following properties.*

1. *If $a \equiv 1 \mod M$, then $\chi(a) = 1$.*
2. *If $\gcd(a,M) = 1$, then $\chi(a)^{\phi(a)} = 1$.*

3.

$$\sum_{a=1}^{N} \chi(a) = \begin{cases} \phi(M) & \textit{if } \chi = \chi_0 \\ 0 & \textit{if } \chi \textit{ is non-trivial.} \end{cases}$$

The proof is omitted here, but can be found in many elementary number theory books, including Chapter 4 of [2]. The first point of Lemma 6.0.3 implies that $\chi(1) = 1$ and $\chi(-1) \in \{1, -1\}$ and brings up the notion of *parity*. A character is said to be "even" if $\chi(-1) = 1$, and "odd" if $\chi(-1) = -1$.

Definition 6.0.4 *The* ***parity*** δ_χ *of* χ *is defined as*

$$\delta_\chi = \begin{cases} 0 & \textit{if } \chi(-1) = 1 \\ 1 & \textit{if } \chi(-1) = -1 \end{cases}.$$

We also introduce this notion of *primitive Dirichlet characters.*

Definition 6.0.5 (Primitive or Induced Dirichlet Characters and Conductors) *The* ***conductor*** C *of a Dirichlet character* χ *modulo* M *is the smallest positive integer such that, for all* a *coprime to M,*

$$\chi(a + C) = \chi(a).$$

If $C = M$*, then the character* χ *is said to be* ***primitive****.*

We note that, in the definition above, if $C < M$, then the character is said to be **induced** from another character χ' modulo C. Notice that such a constant C must be necessary a divisor of M.

Definition 6.0.6 *Given a Dirichlet character* χ *modulo* M *and an integer* n*, the* ***Gauss sum*** *is*

$$\tau(\chi, n) := \sum_{a=1}^{M} \chi(a) e^{\frac{2\pi i a n}{M}}.$$

For clarity, we write $\tau(\chi, 1) = \tau(\chi)$*.*

We now can consider a generalization of the Riemann zeta function called the Dirichlet L-series. For simplicity, we assume χ to be primitive and nontrivial unless otherwise specified.

Definition 6.0.7 *Given a Dirichlet character* χ *modulo* M*, the* ***Dirichlet L-series*** *associated to* χ *is defined as*

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s},$$

where s *is a complex variable with* $\Re(s) > 1$*.*

It follows that $L(\mathbb{1}, s) = \zeta(s)$ because for all integers n, $\mathbb{1}(n) = 1$. Many properties of $\zeta(s)$ can be naturally generalized to the setting of Dirichlet L-series, which are listed below.

Theorem 6.0.8 (Euler Product for Dirichlet L-series) *The Dirichlet L-series $L(\chi, s)$ converges absolutely for $\Re(s) > 1$ and has the Euler product,*

$$L(\chi, s) = \prod_{p \in P} \left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

The Dirichlet L-series also has a symmetry property as well.

Proposition 6.0.9 (Completed L-series Definition and Functional Equation) *Let χ be a non-trivial Dirichlet character modulo M, and we define the completed L-series as*

$$\Lambda(\chi, s) := \left(\frac{M}{\pi}\right)^{s/2} \Gamma\left(\frac{s + \delta_\chi}{2}\right) L(\chi, s).$$

Then $\Lambda(\chi, s)$ satisfies the functional equation

$$\Lambda(\chi, s) = W(\chi)\Lambda(\overline{\chi}, 1 - s),$$

where $W(\chi) = \frac{\tau(\chi)}{i^{\delta_\chi}\sqrt{M}}$.

Proof. The proof is omitted but can be found in many analytical number theory textbooks and papers, such as Chapter 7 of [17], [22], and [13]. □

Similar to how the Riemann zeta function has a connection to Bernoulli numbers and Bernoulli polynomials, the Dirichlet L-series has a connection to a generalization of Bernoulli numbers and polynomials. Let χ be a primitive Dirichlet character modulo M. Then the k-th **generalized Bernoulli number** $B_{k,\chi}$ is defined as

$$\sum_{k=0}^{\infty} B_{k,\chi} \frac{t^k}{k!} = \sum_{a=1}^{M} \chi(a) \frac{te^{at}}{e^{Mt} - 1}. \tag{6.0.1}$$

The generalized Bernoulli numbers satistfy the following property:

$$B_{k,\chi} = M^{k-1} \sum_{a=1}^{M} \chi(a) B_k\left(\frac{a}{M}\right), \tag{6.0.2}$$

where $B_k(x)$ is the k-th Bernoulli polynomial defined in Chapter 4. To see this, we consider

$$\sum_{k=0}^{\infty} B_{k,\chi} \frac{\left(\frac{t}{M}\right)^k}{k!} = \sum_{a=1}^{M} \chi(a) \frac{\left(\frac{t}{M}\right) e^{a\left(\frac{t}{M}\right)}}{e^{M\left(\frac{t}{M}\right)} - 1} = \sum_{a=1}^{M} \frac{\chi(a)}{M} \frac{te^{\frac{a}{M}t}}{e^t - 1} = \sum_{a=1}^{M} \frac{\chi(a)}{M} \sum_{k=1}^{\infty} B_k\left(\frac{a}{M}\right) \frac{t^k}{k!}.$$

By comparing coefficients of the t^k-term, we have

$$\frac{B_{k,\chi}}{M^k k!} = \sum_{a=1}^{M} \frac{\chi(a)}{M} \frac{B_k\left(\frac{a}{M}\right)}{k!},$$

and therefore,

$$B_{k,\chi} = M^{k-1}\sum_{a=1}^{M}\chi(a)B_k\left(\frac{a}{M}\right).$$

Using the above property and Lemma 6.0.3, we have $B_{1,\chi} = \frac{1}{M}\sum_{a=1}^{M}\chi(a)a$.

Remark 6.0.10 *There is also a generalization of Bernoulli polynomials associated with Dirichlet characters. The k-th* ***generalized Bernoulli polynomial*** *$B_{k,\chi}(x)$ associated with a primitive, nontrivial Dirichlet character χ is defined as*

$$\sum_{k=0}^{\infty} B_{k,\chi}(x)\frac{t^k}{k!} = \sum_{a=1}^{M}\chi(a)\frac{te^{(a+x)t}}{e^{Mt}-1}. \tag{6.0.3}$$

To parallel the special values of the Riemann zeta function, we observe the following Theorems.

Theorem 6.0.11 *For a primitive and nontrivial Dirichlet character χ modulo M and a positive integer k, we have*

$$L(\chi, 1-k) = -\frac{B_{k,\chi}}{k}.$$

Theorem 6.0.12 (Special Values of Dirichlet L-series) *For a positive integer k such that $k \equiv \delta_\chi \mod 2$, we have*

$$L(\chi, k) = (-1)^{1+(k-\delta_\chi)/2}\frac{\tau(\chi)}{2i^{\delta_\chi}}\left(\frac{2\pi}{M}\right)^k\frac{B_{k,\overline{\chi}}}{k!}.$$

REFERENCES

[1] L. C. Andrews, *Special functions for engineers and applied mathematicians*. Macmillan Co., New York, 1985.

[2] T. Arakawa, T. Ibukiyama, and M. Kaneko, *Bernoulli numbers and zeta functions*, ser. Springer Monographs in Mathematics. Springer, Tokyo, 2014, With an appendix by Don Zagier.

[3] E. Artin, *The gamma function*, ser. Translated by Michael Butler. Athena Series: Selected Topics in Mathematics. Holt, Rinehart and Winston, New York-Toronto-London, 1964.

[4] R. Askey, "The q-gamma and q-beta functions", *Applicable Analysis*, vol. 8, no. 2, pp. 125–141, 1978.

[5] R. Bellman, *Analytic number theory*, ser. Mathematics Lecture Note Series. Mass.: Benjamin Cummings Publishing Co., Inc., 1980, vol. 57.

[6] R. P. Boas, *Invitation to complex analysis*, Second, ser. MAA Textbooks. Mathematical Association of America, Washington, DC, 2010, Revised by Harold P. Boas.

[7] E. Bombieri, "Problems of the millennium: The Riemann hypothesis", 2000. [Online]. Available: `http://www.claymath.org/sites/default/files/official_problem_description.pdf`.

[8] J. M. Borwein and P. B. Borwein, *Pi and the AGM: A study analytic number theory and computational complexity*. New York: John Wiley and Sons., 1987.

[9] G. F. Carrier, M. Krook, and C. E. Pearson, *Functions of a complex variable*, ser. Classics in Applied Mathematics. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2005, vol. 49, Theory and technique, Reprint of the 1966 original.

[10] K.-W. Chen and M. Eie, "A note on generalized Bernoulli numbers", *Pacific J. Math.*, vol. 199, no. 1, pp. 41–59, 2001.

[11] O. Ciaurri, L. M. Navas, F. J. Ruiz, and J. L. Varona, "A simple computation of $\zeta(2k)$", *Amer. Math. Monthly*, vol. 122, no. 5, pp. 444–451, 2015.

[12] E. Freitag and R. Busam, *Complex analysis*, ser. Universitext. Springer-Verlag, Berlin, 2005, Translated from the 2005 German edition by Dan Fulea.

[13] P. Garrett, “Analytic continuation, functional equation: Examples”, *Preprint*, [Online]. Available: `http://www-users.math.umn.edu/~garrett/m/mfms/notes_c/analytic_continuations.pdf`.

[14] A. Granville, “Harald Cramér and the distribution of prime numbers”, *Scandinavian Actuarial Journal*, vol. 1995, no. 1, pp. 12–28, 1995.

[15] G. A. Jones and J. M. Jones, *Elementary number theory*, ser. Springer Undergraduate Mathematics Series. Springer-Verlag London, Ltd., London, 1998.

[16] M. R. Murty, *Problems in analytic number theory*, ser. Graduate Texts in Mathematics. Springer-Verlag, New York, 2001, vol. 206, Readings in Mathematics.

[17] J. Neukirch, *Algebraic number theory*, ser. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]. Springer-Verlag, Berlin, 1999, vol. 322, Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[18] D. Rockmore, *Stalking the Riemann hypothesis*. Pantheon Books, New York, 2005, The quest to find the hidden law of prime numbers.

[19] E. B. Saff and A. D. Snider, *Fundamentals of complex analysis for mathematics, science, and engineering*. Prentice-Hall, 1976.

[20] C. E. Sandifer, “Euler’s solution of the Basel problem—the longer story”, *Euler at*, vol. 300, pp. 105–117, 2007.

[21] C. J. Sangwin, “An infinite series of surprises”, *Plus. Maths. org. Plus Magazine*, vol. 1, 2001. [Online]. Available: `https://plus.maths.org/content/infinite-series-surprises`.

[22] N. Tanabe, “p-adic L-functions”, 2008. [Online]. Available: `https://math.dartmouth.edu/~ntanabe/p-adic%20L-functions.pdf`.

[23] D. Wilkins, “On the number of prime numbers less than a given quantity”, 1998. [Online]. Available: `http://www.claymath.org/sites/default/files/ezeta.pdf`.