ASPECTS OF ALEXANDER DUALITY

A Thesis

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Abstract

Alexander Duality is a crucial result in Algebraic Topology formally relating a topological object to its complement when embedded in a sphere. In this paper we modernize J. W. Alexander's proof of the Jordan Arc Theorem and Jordan Curve Theorem. These easily generalize to his original proof of Alexander Duality, which we also modernize. We then dive into a study of Alexander's Horned Sphere, an object invented by Alexander to disprove an earlier conjecture of his that no wild sphere embeddings existed in E^3. We specifically study the non-trivial crumpled cube in the complement of Alexander's Horned Sphere through the lens of homotopy groups and geometric group theory, and show that it is equivalent to a thickened infinite height genus 1 grope. We also present a potentially novel visualization of this grope in context with Alexander's Horned Sphere. Finally, we explore variant horned spheres and present a particularly interesting case, Lindsey's Horned Sphere, which has a significantly more complex crumpled cube complement.

Acknoledgements

I have been interested in Alexander Duality ever since I took Professor Doyle's excellent open-ended class on geometry and topology, 'Shape of Space' in my sophomore year. I approached him after class and we discussed how inserting 'negative space' balls inside a sphere changed its Euler number (I didn't know what homology was at that point) in a very predictable way. He told me I had just stumbled across the core idea of Alexander duality; a relationship between an object and its complement. As the thesis progressed and we studied Alexander's Horned Sphere we began to understand how beautiful of an object it was, despite the often disturbing-looking drawings people made of it, and thus shifted the scope of the thesis to focus more on Horned Spheres.

I would like to give a massive thank you to my advisor Professor Peter Doyle who not only provided the initial inspiration for the topic but helped me at every step of the process. His insatiable curiosity for interesting mathematical objects and willingness to deep dive into anything, even something usually thought of only as a pathological object, such as Alexander's Horned Sphere, has inspired me to focus more on the beauty and elegance of math rather than purely using it as a means to an end. This has been an invaluable opportunity for me and I cannot put into words how much Professor Doyle's help and support has meant to me and furthered me as a mathematician. I would also like to thank the rest of the Department of Mathematics at Dartmouth College for providing me with an excellent education in the introductory elements of higher math.

I would also like to thank Professor Michael Freedman, who provided his invaluable insights into the complement of the Horned Sphere and introduced us to the concept of a grope, which became central to this thesis. On top of that, I'd like to thank Professor Michael Starbird for giving more insights into wild surfaces in E^3.

I also owe large inspiration and gratitude to Professor Kathryn Lindsey of Boston College, whose variant horned sphere became central to the final portion of my thesis. I'd like to thank Professor Sabetta Matsumoto of Georgia Tech as well, for corresponding with us over email about building a physical model of the Horned Sphere.

Finally, I am very grateful for the support of my friends and especially my family throughout my undergraduate years. I could not have achieved anything without their support.

Contents

Chapter 1

Introduction

Alexander Duality is a well-known result of Algebraic Topology, usually proved with diagram chasing as a corollary of Poincaré duality, as in [Hat02]. However, When J. W. Alexander first came up with Alexander Duality, homology, cohomology, and category theory did not exist. Alexander's proof, just over 100 years old now, in fact laid the groundwork for the entire field of cohomology.

Of course, Alexander Duality did not emerge from a vacuum; instead it was an easy generalization of another beautiful proof of Alexander's: a proof of Jordan's Curve Theorem. Jordan's Curve Theorem (JCT) seems straightforward to non-mathematicians: a circle separates the plane into two components. However it was surprisingly difficult to prove and there still exists controversy about whether Jordan's original proof is valid. Alexander approached the problem with a highly generalizable approach that introduced chains (an older analogue of CW-complexes) and unequivocally proved the JCT.

Just two years later, he published his description and proof of Alexander Duality in arbitrary dimension, at the time stated in terms of connectivity numbers

(Betti numbers + 1), whereas the modern form is stated in terms of homology and cohomology groups.

The next three chapters of this thesis will modernize Alexander's proof of the Jordan Curve Theorem, Jordan Arc Theorem (JAT) in higher dimensions, and Alexander Duality in the case of spheres, respectively. Then, we will discuss Alexander's Horned Sphere, a wild embedding of the 2-sphere in E^3. Finally we will conclude by discussing the wide world of variant Horned Spheres.

This thesis, and mathematics as a whole, owe much to J.W. Alexander and his contemporaries such as Oswald Veblen. My hope is that the first few chapters of thesis will communicate some of Alexander's genius, while the last few expand into the world of wild surfaces that are rarely studied since the prolific publications of James W. Cannon in the mid-late 20th century.

Chapter 2

The Jordan Curve Theorem

Section 2.1

Alexander's Approach to the JCT

J.W. Alexander provided a fairly straightforward proof of the Jordan Curve Theorem, that a simple closed curve divides the plane into two components, using the Jordan Arc theorem and other intermediary lemmas, in [Ale20]. His argument is based on the property of systems of line segments forming polygonal shapes. In this paper we aim to recount his proof using more modern terminology and ideas, while maintaining the same structure and ideas that Alexander proposed.

Alexander's proof is of particular note because it generalizes to higher dimensions and implies the basic ideas of homology and Alexander duality, as well as providing one of the first largely accepted proofs of the Jordan Curve theorem, whose history is fraught with controversy.

Section 2.2

Chains

We will use the idea of chains to represent generalized polygonal shapes which will simplify our proof. A chain is a finite number of non-intersecting edges (line segments or rays) and vertices (the points at the end of each edge) where each vertex must have an even number of edges terminating at it. Note a chain does not have to be connected.

Any two points on a chain that can be connected through the chain can be connected through at least two different paths through the chain which only intersect in a finite number of points. This is because if one path exists, deleting that path will leave two vertices with an odd number of edges terminating at it. But within any connected group, the total degree of all the vertices must be even, hence the vertices still belong to the same connected piece and can be joined.

Like a polygon has an interior and exterior, chains also have two 'sides', though each side need not be connected. Any point in a plane not in a chain k can be can be separated into two distinct classes relative to the chain k, and we call those classes 'sides'. See Figure 2.1 for a visual example, and the appendix for more information on how those sides are defined.

Lemma 2.1. *Given any two chains k_1 and k_2, the set of points on given sides of both k_1 and k_2 can be subdivided into finitely many convex regions.*

Therefore, the set is bounded by a chain composed of the sum, modulo 2, of the boundaries of the convex regions. By adding this chain with the original chain, we can obtain another new chain.

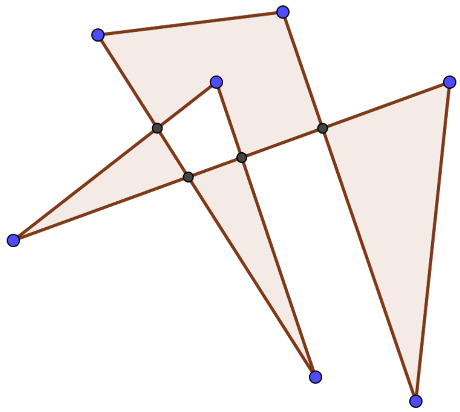


Figure 2.1: A chain, with the two sides marked. One side is shaded orange, the other is left blank (white). Note the side shaded white also includes all the space outside the shape.

Section 2.3

Jordan's Theorem

Definition 2.2. A *region* is a open, path-connected set of points.

Alexander's original paper uses open triangles around points in a region, but semantically that is equivalent to open balls, and by the equivalence of p-metrics, open squares as well.

A corollary to this definition is that any two points Y and Z in a region can be connected within the region by a broken line l with a finite number of points in common with any preassigned finite system of lines. This follows from the Heine-Borel theorem and path-connectedness.

Definition 2.3. A *simple arc* is the image of an injective continuous function $f : [0, 1] \to \mathbb{R}^2$.

Lemma 2.4. *Let ACB be a simple arc passing through C, ending at points A and B, and let Y and Z be any two points in the plane not on the arc ACB. Then if the points Y and Z are not separated by either sub-arc AC or BC, then they are not separated by ACB.*

Since it is not separated by either sub-arc, there are broken lines a and b connecting Y to Z such that a does not intersect AC and b does not intersect BC.

Then, by the corollary to the definition of a region, b can be chosen to meet a at only a finite number of points. Then, we can combine a and b to form a chain k.

Now, consider the closed subset X of BC such that they either intersect k or are on the opposite side of k as C. Each of these points can be encased in an open square, each of which can be constructed to neither meet nor enclose any points on AC or b. Since X is closed, we can enclose all of X within a finite number of these open squares, by the Heine-Borel theorem.

Now, let us add modulo 2 to k the boundaries of the union of these open squares to create a new chain k', which still contains all of b, and a supplementary piece a', made of arcs which neither meet nor end on the arc ACB.

Therefore, since Y and Z are on the chain k', they can be joined by a broken line which includes the piece a' which does not meet the arc ACB.

See Figure 2.2 for the diagram of this construction in a simple case.

Theorem 2.5. *Jordan Arc Theorem: The points of the plane not on a simple arc AB do not form more than one connected region.*

We will show any two points Y and Z not on the arc can be joined by a path which does not meet the arc AB.

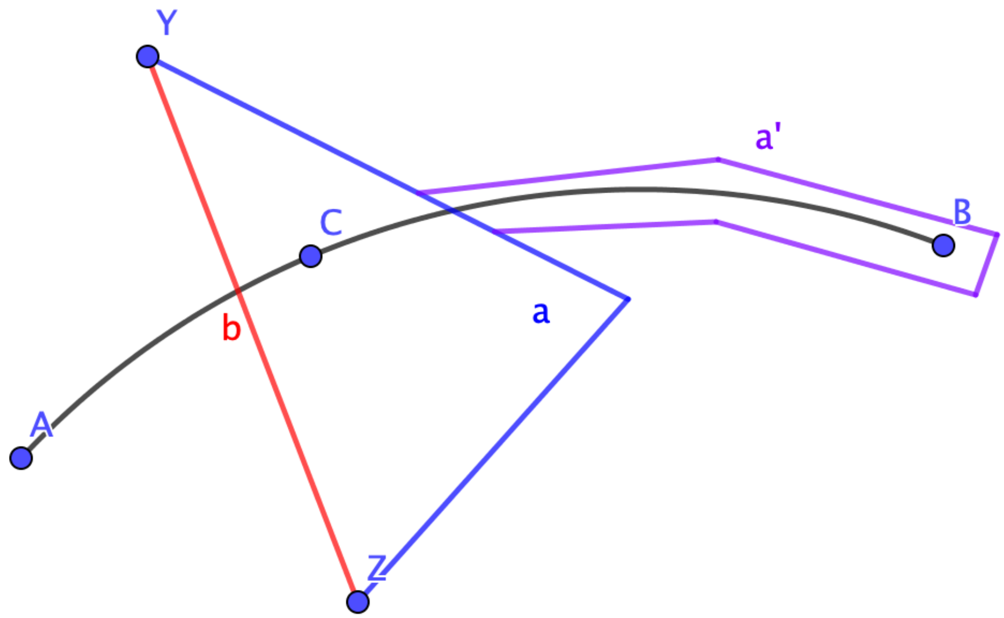


Figure 2.2: Example of Lemma 2.4 in a simple case. Imagine a' to be the union of the finite open squares around points in X as described in the proof. $k = a + b$, $k' = a' + b +$ the required extra parts of a to complete the chain. k' clearly avoids BC, and therefore a' constructs a path from Y to Z that avoids all of ACB.

Take any point C on the arc AB. We can always construct an open square around C which does not include Y and Z. Within this open square, we can make an sub-arc of AB which passes through C, does not separate the points Y and Z, and ends at C only if C is A or B.

Thus the arc AB can be covered by a set of overlapping sub-arcs, and by Heine-Borel, by a finite number of those sub-arcs, none of which separate Y and Z. Then, by inductively applying Lemma 2.4 to each of these finite number of sub-arcs, the arc AB constructed out of their union cannot separate Y and Z either. Thus Y and Z are in the same region.

Definition 2.6. A *simple closed curve* is the image of an injective continuous function $f : S^1 \to \mathbb{R}^2$.

Theorem 2.7. *The points of the plane not on a simple closed curve do not form more than two connected regions.*

We will show for any three points X, Y, Z not on a simple closed curve, at least two of them are in the same region.

Let A, B, C be three distinct points on the simple closed curve. Then, by the Jordan Arc Theorem (Theorem 2.5), the points Y and Z, Z and X, and X and Y can be joined by three broken lines, a, b, and c respectively, which do not meet the arcs CAB, ABC, and BCA respectively. Also, a, b, and c can be chosen so that they each intersect each other only at a finite number of points. Then they form a chain $k = a + b + c$. See Figure 2.3 for a diagram of this chain.

Now by the pigeonhole principle, at least two of A, B, C must be on the same side of k. Assume without loss of generality it is A and B. Then by the proof to Lemma 2.4, substituting BCA for the arc AC and the broken line ab for b, X and Y can be connected by a broken line which does not meet the curve. Therefore, they are in the same region, so there are at most two regions.

Theorem 2.8. *The points of the plane not on a simple closed curve form at least two connected regions.*

Choose any two points A,B on the curve and denote the two arcs separated by those points as AB and BA. Then any line l which separates the points A and B meets the arcs AB and BA in two closed sets of points respectively, X and Y.

Every point in X is interior to some interval of line l which contains no point of the arc BA, so by Heine-Borel theorem, the entire set can be covered by a finite set of

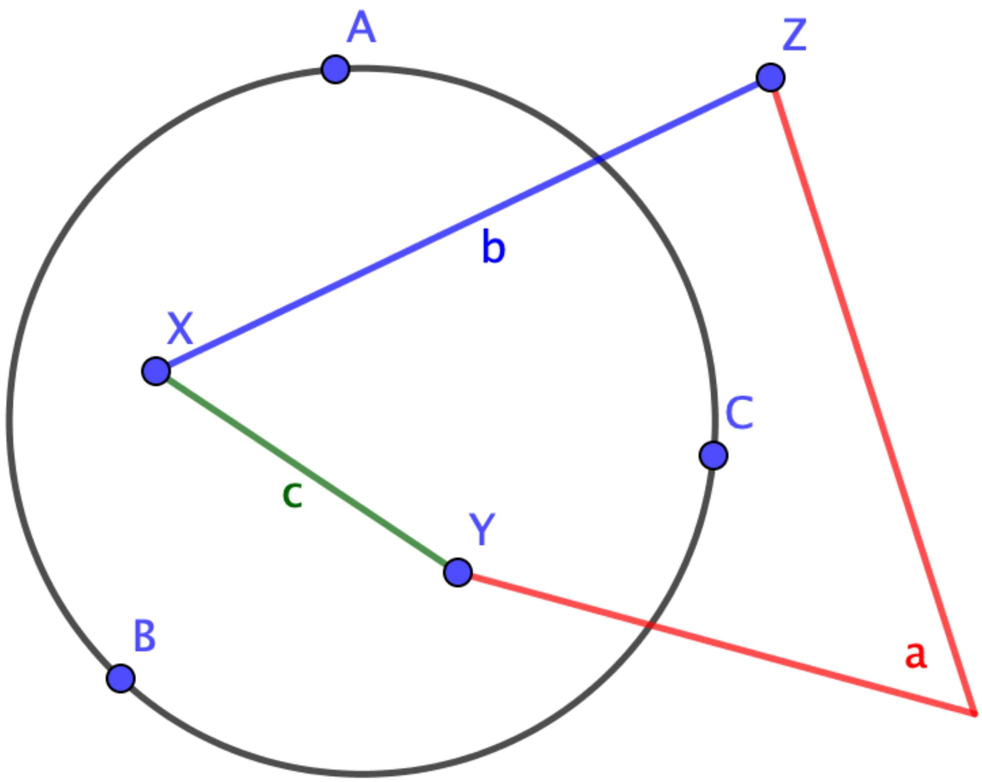


Figure 2.3: Given the curve ABC and the points X, Y, Z, we construct the chain $k = a + b + c$ which has A and B on one side (the 'exterior') and C on the other side (the 'interior'). Then, using the same logic as the proof to Lemma 2.4, we can create a broken line connecting X and Y that does not touch the curve (such as c), therefore they are in the same region.

such intervals. Furthermore, we can arrange it so no such intervals overlap or touch. We can prove that the end points of this set of intervals i are not all within the same region by showing that any broken line connecting the end points will intersect the curve.

Assume by contradiction that there is a system of broken lines connecting the endpoints of the intervals i such that none meets the line l or another broken line in the system in more than a finite number of points. Then this can be combined with the intervals i to form a chain k. If we add l to k, we get a another chain k' made of broken lines combined with segments s of the line l complementary to i. Then, by definition of sides of a chain, either k separates A and B, or k' separates A and B.

If k separates A and B, it must meet BA, but BA cannot meet any of the intervals i so must meet one of the broken lines. Similarly if k' separates A and B, AB must meet one of the broken lines. So no matter what, the curve meets one of the broken lines, which is a contradiction therefore the intervals do not all belong to the same region. So there are at least two regions.

Theorem 2.9. *The points of the plane not on a simple closed curve form exactly two regions.*

Follows from the last two theorems.

Corollary 2.10. *A point Z not on a simple closed curve may be connected to any arc of the curve AB by a broken line z which, except for one end point, lies wholly within the same region as Z.*

Let Y be any point on the opposite side of the curve from Z. Then, by the Jordan Arc Theorem (Theorem 2.5), Y and Z may be connected by a broken line which passes through AB but not BA. This broken line must include the desired broken line z connecting Z to AB.

Corollary 2.11. *In the neighborhood of any point A on a simple closed curve there are points from each side of the curve.*

This follows from the previous corollary since Z can be chosen on either side of the curve and the arc AB can be made arbitrarily small.

Chapter 3

Generalizing Jordan's Theorems

How do Jordan's Arc and Curve Theorem Generalize to higher dimensions? In fact, there is quite an obvious way to state these generalized theorem: simply use the higher-dimensional analogues of an arc and a circle, an i-disc (D^i) and an i-sphere (S^i) respectively. A fun and easy exercise is generalizing the JAT and JCT backwards to 1-dimensional objects, where one can note that the relationship still holds and the proof mirrors the 2D proof in interesting ways.

For 1D JAT, one must show that the complement of point p immersed in S^1 has one connected component: this is obvious since $S^1 - \{p\} = \mathbb{R}^1$. Interestingly, the immersion in is S^1, the compactification of $\mathbb{R}^1$, not $\mathbb{R}^1$ itself. This is required to maintain the same spirit has the JAT: if it was in R^1, a point would separate its complement into two distinct components, whereas in S^1 it does not. This is one demonstration of why Alexander Duality is always done immersed in the compactification of $\mathbb{R}^n$, an n-sphere and not R^n directly. For 1D JCT, one must show the complement of a 0-sphere (two points $\{p, q\}$) immersed in S^1 must have two connected components; which is again trivial. These are restatements of T^0 and X^0 with $n = 1$, respectively.

However, in this section we will be focusing on arbitrary dimension. Before we can formally state and prove these theorems, we must cover some background information.

Section 3.1

Background Information

Several logical extensions of ideas we used in the 2D JCT and JAT Proof will be used in the generalized proof. One of the most important is the continued use of chains. The chains of arbitrary dimension are slightly different than the simple piecewise linear ones defined in the 2D proof, instead defined through selectings cells from planar subdivisions of a sphere, but the spirit remains the same.

The important information to note is that 'chain' is for most intents and purposes interchangeable with 'CW-complex', and in modern formulations of Alexander Duality, it is even more general than that. Note that discs and spheres of any dimension are themselves chains.

3.1.1. Special Types of Chains

There are certain subclasses of chains and terminology we will use to describe chains that will make this proof significantly more concise.

An *i-chain* will be a chain that is built only out of i-cells and their boundaries. For example a graph would be a 1-chain, but a graph with an extra disconnected point added would not be a 1-chain, since that extra disconnected point is not a 2-cell and is not in the boundary of any 2-cells in the chain.

A *cellular* i-chain is an i-chain consisting only of a single i-cell: in other words, it is an i-disc D^i.

Any i-chain is said to be the sum modulo 2 of all the i-cells that make it up. This makes sense because the boundary between two touching cells is no longer a boundary of the chain, and since it gets added to itself (one for each of the two touching cells), modulo two it goes away. Using this, we can express an i-chain as a sum of i-cells:

$$K^i = K_0 + K_1 + \cdots + K_n$$

and the sum of two chains can be calculated by adding their cell-sum forms modulo two. For the equation above, and all other following cell-sum equations, assume they are done modulo 2.

An i-chain is *closed* if each of its $(i-1)$ cells belong to an *even* number of its i-cells, otherwise it is called *open*. An open chain will have a boundary, and it boundary will be exactly the $(i-1)$-cells that belong to an odd number of i-cells of the chain. We will denote the $(i-1)$-chain ∂K^i as the boundary of the i-chain K^i. If $\partial K^i = 0$, then K^i has no boundary and is a closed chain. In homology terminology, a closed chain is a cycle, and an open chain is not. Since we are using modulo 2, a 0-chain (collection of disjoint points) will be open or closed if the number of points is odd or even, respectively.

It can be easily shown $\partial^2 X = 0$ for any X, so any boundary is automatically closed. It is also intuitive that the sum of two closed chains is itself closed.

For more information on chains and their properties, see [Ale22].

3.1.2. Betti Numbers and Connectivity Numbers

The *connectivity number* is an old-fashioned relative of the Betti Numbers, which are themselves an old-fashioned version of Homology groups. There is a connectivity number in each dimension for a chain. We will denote the i-th connectivity number of a chain with the number R^i.

Connectivity numbers and Betti number have an important relation that makes them easier to understand. With $\bar{b_i}$ as the reduced i-th Betti number of some chain:

$$R^i = \bar{b_n} + 1$$

Note that R^0 is the number of separate connected components of the chain. For a description and more concrete proof of how connectivity numbers relate to Betti numbers and actual topological information, see [Ale22].

Also note that if the i-th connectivity number of a chain is 1, then any closed i-subchain is the boundary of some open $(i+1)$-subchain; we say then that the i-chain *bounds*. From a modern perspective, this is intuitive because if the Betti number in the i-dimension is 0, then there are no (excluding torsion) nontrivial i-cycles, which is to say no closed i-chain that is not the boundary of some higher dimensional chain. Thus all closed i-chains must bound some higher dimensional chain.

Due to some intricacies of Alexander Duality, some of which even emerge in the 1-dimensional case as discussed before, all of the following chains will immersed in S^n, the compactification of $\mathbb{R}^n$.

Section 3.2

Generalizing Jordan's Arc Theorem

We can now show a generalized version of JAT.

Theorem 3.1. *T^i. Let C^i be a cellular i-chain (an i-disc) immersed in the n-sphere*

S^n. Then the connectivity numbers of the complement $S^n - C^i$ are all 1. In other words, every closed chain L^k of $S^n - C^i$ bounds.

We will prove this using induction. The proof of T^0 is trivial: if $i = 0$, C^0 is a single point, and thus no matter what n is, every closed chain in $S^n - C^0$ bounds since you can just perturb an open chain bounded by the closed chain to avoid C^0. (See [Ale22] for a slightly more rigorous proof of this).

For the inductive step, however, we will need the help of more machinery. It is notable that the heavy lifting of the JCT proof is done by the lemma allowing us to 'combine' two arcs without separating the residual space into two complements (Lemma 2.4). Then, as shown in the previous chapter, by using compactness we can easily expand this to the Jordan Arc Theorem, and can use that to prove that there are at least 2 regions in the complement of a closed curve, which is the difficult part of proving the JCT.

Therefore, it is not surprising that in the generalized case of higher dimensions, and also the true generalization of Alexander Duality, the lemma that does the 'heavy lifting' allows us to combine 2 discs without separating the residual space:

Lemma 3.2. *U^i. Let the cellular i-chain C^i (an i-disc) be subdivided into two cellular i-chains A and B, meetings in a cellular $(i-1)$-chain C^{i-1}. Then every k-chain L^k of $S^n - C^i$ which bounds in both $S^n - A$ and $S^n - B$ must also bound in $S^n - C^i$.*

There are two main cases: $k = n-1$, and $k < n-1$. The first case is trivial, since any closed $(n-1)$-chain bounds two regions of S^n, and all of $C^i = A + B$ must be in the same region (once again, see [Ale22] for a rigorous explanation). The second case is the interesting one.

Assume $k < n-1$, so there exists chains in S^n of dimension $k+2$. Then there are, by the statement of the lemma, two open $k+1$ chains X and Y such that $\partial X = L^k$ in the complement of A ($S^n - A$), and $\partial Y = L^k$ in the complement of B ($S^n - B$). Since they have the same boundary, the chain $Z = X + Y$ has no boundary, so $\partial Z = 0$ means Z is a closed chain.

This is exactly the higher dimension version of the step in the proof of Lemma 2.4 where we combine a and b into the chain k.

We will assume that X and Y meet B and A respectively, otherwise the theorem is automatically true.

Then by the inductive use of theorem T^{i-1}, there exists some open ($k+2$)-chain M such that $\partial M = Z$ in the complement of C^{i-1}. Note that this chain will potentially intersect A and B. If it does, it will do so in mutually exclusive closed sets of points.

Now due to compactness, we can take $\bar{M}$ to be the close of the points of M that intersect A and not B. Now we can take the modulo 2 sum of $M + \bar{M}$; this effectively cuts out the points from M where it would touch A. Consider the boundary of this:

$$\partial(M + \bar{M}) = \partial M + \partial\bar{M} = (X + Y) + \partial\bar{M} = (Y + \partial\bar{M}) + X \qquad (\text{in } S^n - C^{i-1})$$

Of course, since $\partial^2 = 0$, it is known that $\partial((Y + \partial\bar{M}) + X) = 0$. But therefore, $\partial(Y + \partial\bar{M}) = \partial X = L^k$, since modulo 2 they must cancel to zero.

And we already know that $Y + \partial\bar{M}$ meets neither A nor B, because Y is designed not to meet B, and $\partial\bar{M}$ never touches B (because $\bar{M}$ doesn't) and touches A in all the same places that Y does, and no others, so modulo 2 they cancel out. Therefore, $Y + \partial\bar{M}$ is in $S^n - C^i$. So the chain $(Y + \partial\bar{M})$ has boundary L^k and lies in the complement of C^i, proving the lemma.

Proving T^i then follows naturally from a contradiction argument (see [Ale22] for a more rigorous explanation): if T^i is false for some disc C^i, then by the contrapositive of U^i it must be false for one of two halves of that disc A or B. Repeating this infinitely, you create a sequence of smaller and smaller discs that ends on a single point that must violate T^0. But we have shown in the base case that T^0 works for a point, which is a contradiction.

One can notice that each step of the proof of U^i is the direct higher-dimensional analog of the steps used in proving Lemma 2.4 in 2D. The ease with which that proof can be generalized to higher dimensions, then to full Alexander Duality soon after, which itself implies the entire machinery of cohomology, shows how powerful Alexander's original ideas on the Jordan Curve Theorem were.

Chapter 4

Generalizing the Jordan Curve Theorem

Corollary 4.1. W^i. *Let C be the sum of C^{i-1} be the intersection of two closed sets of points A and B. Then every closed k-chain L^k when $k < n-1$ of $S^n - C$ which bounds a chain L_A^{k+1} in $S^n - A$ and a chain L_B^{k+1} in $S^n - B$ must also bound a chain in $S^n - C$ provided the chains L_A^{k+1} and L_B^{k+1} may be chosen so that $(L_A^{k+1} + L_B^{k+1})$ bounds in $S^n - C^{-1}$. Moreover, the corollary is valid even if $k = n-1$ unless C^{i-1} is the null set.*

Proved by induction on i. This is a generalized form of Lemma U^i and uses effectively the exact same proof as it, just with some relabeling.

Theorem 4.2. X^i. *(Alexander duality in the special case of spheres) Let C^i be an i-sphere immersed in an n-sphere S^n. Then the connectivity numbers R^s of C^i are related to the connectivity numbers of the complement $S^n - C^i$ by the equations*

$$R^i = \bar{R}^{n-i-1} = 2, \quad R^s = \bar{R}^{n-s-1} = 1 \ (s \neq i)$$

This theorem states, in other words, that there exists exactly one independent closed non-bounding chain in $S^n - C^i$ which will be of dimension $(n - i - 1)$. This chain will be said to *link* the i-sphere C^i.

We will show certain specific cases of this theorem before moving on to a full proof.

Section 4.1

Jordan Curve Theorem Reformulated

The Jordan Curve theorem is a corollary of X^1, since for a 1-sphere (circle, $i = 1$) in 2D ($n = 2$), showing that $\bar{R}^{n-i-1=0} = 2$ is equivalent to saying the complement of the immersed circle has two components.

Due to the inductive nature of the proof of X^i, we will first have to prove X^0.

4.1.1. X^0

In the case $i = 0$, the 0-sphere C^0 is just a pair of points, so the theorem is trivial since the $(n-1)$-chain linking C^0 is any closed $(n-1)$ chain L such that one of the two points lies in each of the two chains bounded by L, such as a $(n-1)$-sphere around one point which doesn't contain the other. All chains of lower dimension bound in the complement.

4.1.2. Back to Jordan Curve Theorem

Let us subdivide the sphere C^1 into two cellular 1-chains (1-discs) A and B which meet in a $(i-1) = 0$-sphere C^{i-1}. Note we will be focusing on the case $n = 2$.

By induction, there is a chain L^1 of $S^n - C^0$ which links C^0, and necessarily meets A and B in mutually exclusive, closed sets of points: because if it failed to meet A, for example, it would bound in $S^n - A$ by theorem T^i, and therefore in $S^n - C^0$ as well since $C^0 \subset A$, contrary to the hypothesis that L^1 links $S^n - C^0$.

Therefore, the chain L^1 can be written as a sum of two open chains (similar to proof of U^i) such that

$$L^1 = L^1_A + L^1_B$$

Lying in $S^n - A$ and $S^n - B$ respectively, and having a common boundary, a chain L^0 in $S^n - C^1$. Then the chain L^0 links C^1. See Figure 4.1 for a diagram of the described situation.

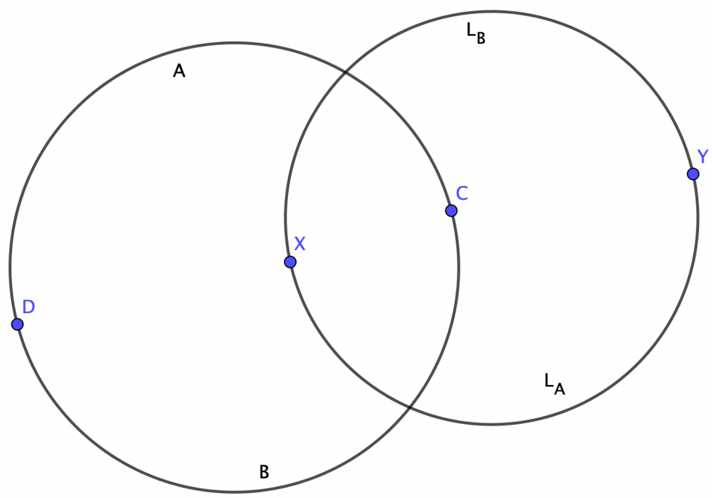


Figure 4.1: Here we see the linking diagram of the immersed circle $C^1 = A + B$. In this figure, $C^0 = \{C, D\}$. L^1, which is guaranteed to exist and link C^0 by the inductive step, $= L_A + L_B$, and the linking chain $L_0 = \{X, Y\}$. L_0 having two points demonstrates that there are at least two components in the complement of the circle.

If it did not link C^1, then there would be an open chain $\bar{L}^1$ bounded by L^0 in $S^n - C^1$. Then either $\bar{L}^1 + L^1_A$ or $\bar{L}^1 + L^1_B$ would have to link C^0, because their sum $\bar{L}^1 + L^1_A + \bar{L}^1 + L^1_B = L^1_A + L^1_B = L^1$ does. Assume without loss of generality that

$\bar{L}^1 + L^1_A$ links C^0. This is a contradiction, because to link C^0, the chain must intersect A, but both $\bar{L}^1 \in S^n - C^1 \subset S^n - A$ and L^1_A is specifically constructed to avoid A. Therefore L^0 links C^1. See Figure 4.2 for a diagram of the why no $\bar{L}$ exists in this situation.

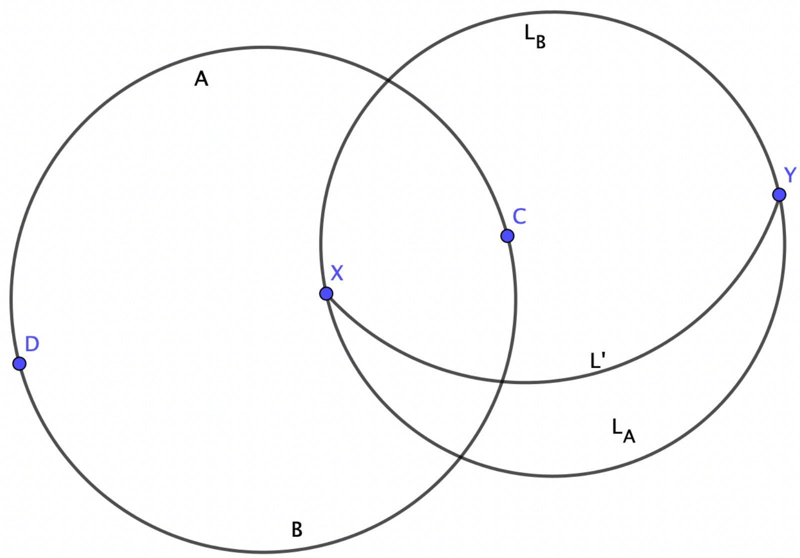


Figure 4.2: Here we see why $L^0 = \{X, Y\}$ must link C^1: if it did not, there would exist an open chain $\bar{L}^1$ (labelled L' on the diagram), which does not intersect A or B, such that either $L' + L_B$ or $L' + L_A$ linked C^0 (in this case, $L' + L_B$ links C^0). However that is not possible since however you draw L' to do so it must intersect $C^1 = A + B$.

Finally to show the independence of L^0. Assume there is another linking chain M^0. Then there is a closed chain $M^1_A + M^1_B$ defined relative to M^0 in the same manner as $L^1_A + L^1_B$ are relative to L^0. Then there is a chain associated with $L^0 + M^0$ as well:

$$(L^1_A + M^1_A) + (L^1_B + M^1_B)$$

which cannot link C^0. Then by corollary W^i, $L^0 + M^0$ bounds in $S^n - C^1$ and M^0

is dependant on L^0, thus L^0 is an independent linking chain. See Figure 4.3 for a diagram of the why L^0 is independent.

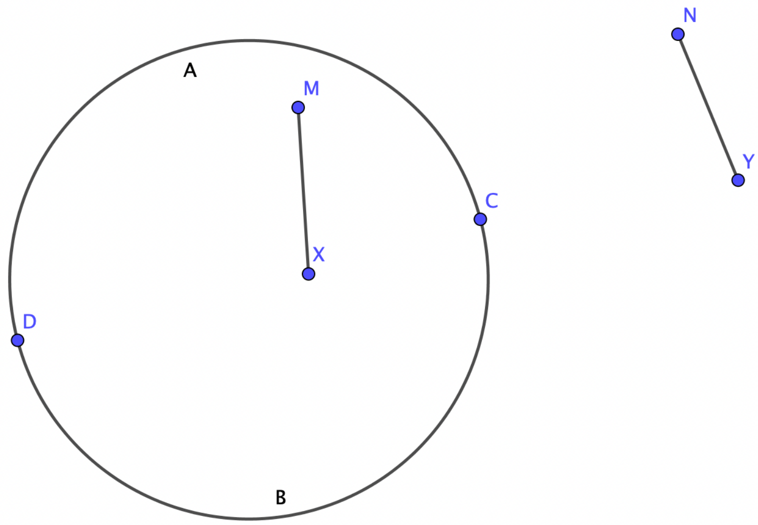


Figure 4.3: In this figure we image another chain linking $C^1 = A + B$: the chain $M^0 = \{M, N\}$. We can see that the chain $M^0 + L^0 = \{M, N, X, Y\}$ bounds in $S^2 - C^1$, as shown in the diagram, and therefore their sum cannot link. So L^0 is independent.

Section 4.2

Arbitrary Dimension

We have explicitly proved X^1 because its 2-dimensional nature makes visual aids easier, but the inductive case for an arbitrary dimension i is easy to extract from what is written above: simply replace any time I have a specific dimensional value with i, k or n (each ± 1 or 2) as appropriate.

Section 4.3

Full Alexander Duality

Proving full Alexander Duality in the modulo 2 system we've been operating in is both unnecessary, as it in done in [Ale22], and beyond the scope of this thesis. However, the proof follows a very similar path of logic to the proof of X^i, which is itself a specific case of general Alexander Duality.

For the remainder of this thesis we will be using the full, modern formulation of Alexander Duality, which is as follows: For a compact, locally contractible, nonempty proper subspace $X \subset S^n$, then for all i:

$$\tilde{H}_i(S^n \setminus X) \cong \tilde{H}^{n-i-1}(X)$$

where $\tilde{H}_i$ is the ith reduced homology group and $\tilde{H}^j$ is the jth reduced cohomology group. A nice proof of this using algebraic topology can be found in Hatcher ([Hat02]).

Chapter 5

Alexander's Horned Sphere

Alexander's Horned Sphere is a interesting wild object that was invented by Alexander to disprove his own conjecture (the 3D analog of the Schoenflies theorem): it is an example of an immersion of the 2-sphere in S^3 such that its interior and exterior are *not* both homeomorphic to 3-discs. See Figure 5.1 for a fairly standard embedding into 3-space.

The construction of the horned sphere ends in a set of wild points that form a Cantor set, which is clearly visible in Figure 5.2.

Section 5.1

What makes the Horned Sphere special

The interesting part of the Horned Sphere is its complement, specifically the exterior component, which is what makes it a counterexample to the 3D analogue of the Schoenflies theorem.

We will define Alexander's Horned Ball (AHB) to be Alexander's Horned Sphere (AHS) with the simply connected interior filled in. This is homeomorphic to a ball

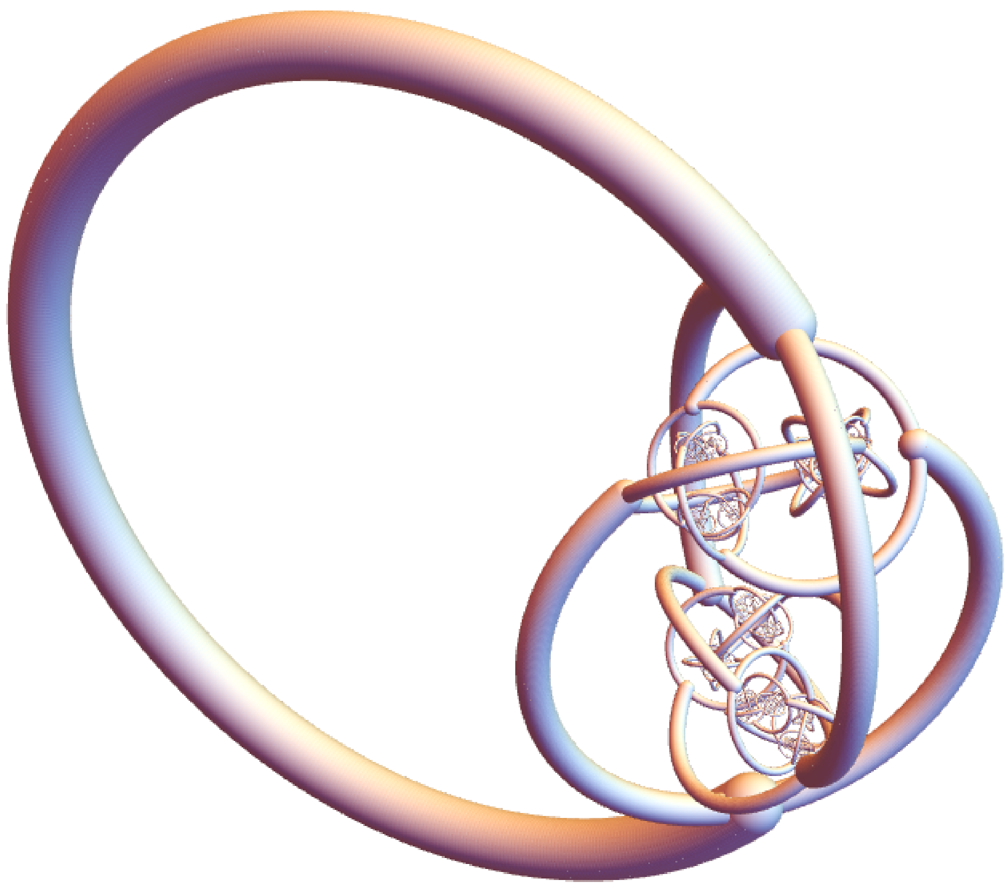


Figure 5.1: An embedding of Alexander's Horned Sphere

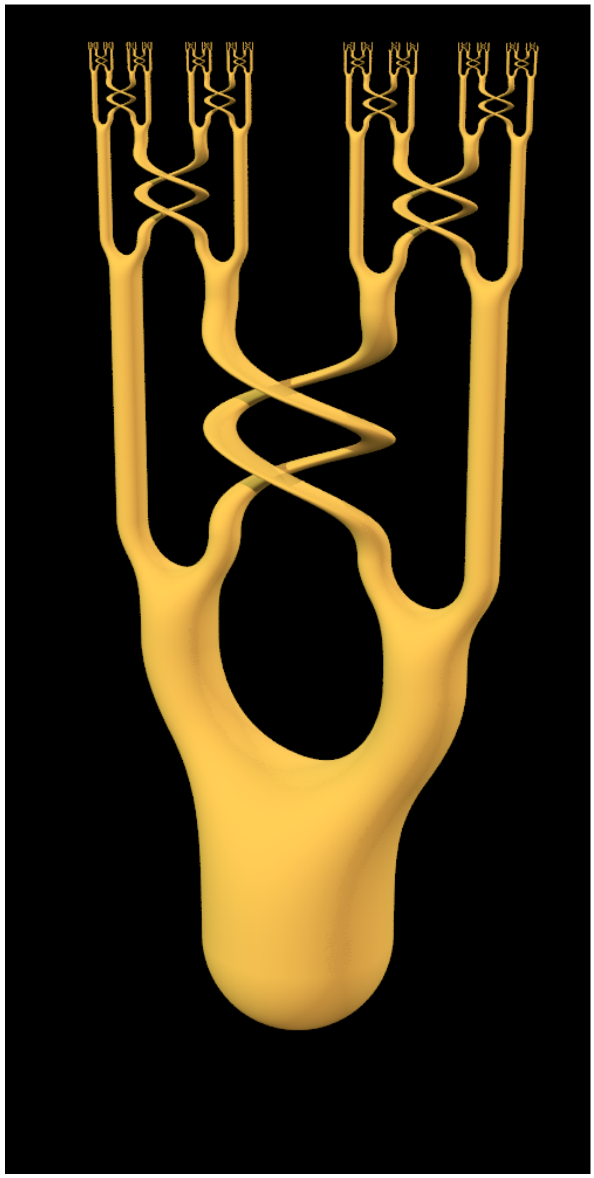


Figure 5.2: An embedding of Alexander's Horned Sphere showing the Cantor set of wild points at the top. Reprinted from [Ché19], originally designed by Bob Edwards.

i	0	1	2
$H^i(AHB)$	$\mathbb{Z}$	0	0
$\tilde{H}^i(AHB)$	0	0	0
$\tilde{H}_i(S^3 - AHB)$	0	0	0
$H_i(S^3 - AHB)$	$\mathbb{Z}$	0	0
$b_i(S^3 - AHB)$	1	0	0

Table 5.1: Cohomology and homology groups of the Alexander Horned Ball and its complement, respectively.

(D^3). We will be focusing on describing the wild complement: $S^3 - AHB$, the non-simply connected exterior of the AHS.

This complement is automatically what known as a crumpled cube:

Definition 5.1. Take some object $X \subset S^{n+1}$ which is homeomorphic to S^n. Let U be a component of $S^n - X$. Then $C = Closure(U)$ is a *crumpled cube* with boundary X. Note that an n-crumpled cube may not necessarily be the same as an n-cube.

For example, when X is just the standard embedding of a 2-sphere in S^3, both crumpled cubes it generates are simply 3-discs, and homeomorphic to 3-cubes. However, when X is Alexander's Horned Sphere, one crumpled cube (the interior one) is just a disc, but the other (the exterior $S^3 - AHB$) is not homeomorphic to a disc, and is the subject of our study in this chapter. See [Can78] for more information on crumpled cubes.

Since the Horned Ball is homeomorphic to a 2-ball, we can use Alexander Duality to examine the homology groups and Betti numbers of the crumpled cube ($S^3 - AHB$). As we can see in Table 5.1, the homology of the complement tells us nothing: through homology and cohomology alone, the complement of Alexander's Horned Ball is indistinguishable from a disc. So instead of relying on the nice, abelian homology groups, we must instead turn to the fundamental group to find our answer. See

[Hat02, p172] for a formal construction of the Horned Sphere and the fundamental group of its complement.

The fundamental group can be determined by applying Wirtinger's process for knots at various levels of recursion, then taking the direct limit of the groups as you add more and more levels of the AHS. For example, we can see the result with one level of recursion in Figure 5.3, which shows that the bottommost grasping curve is a commutator of higher ones.

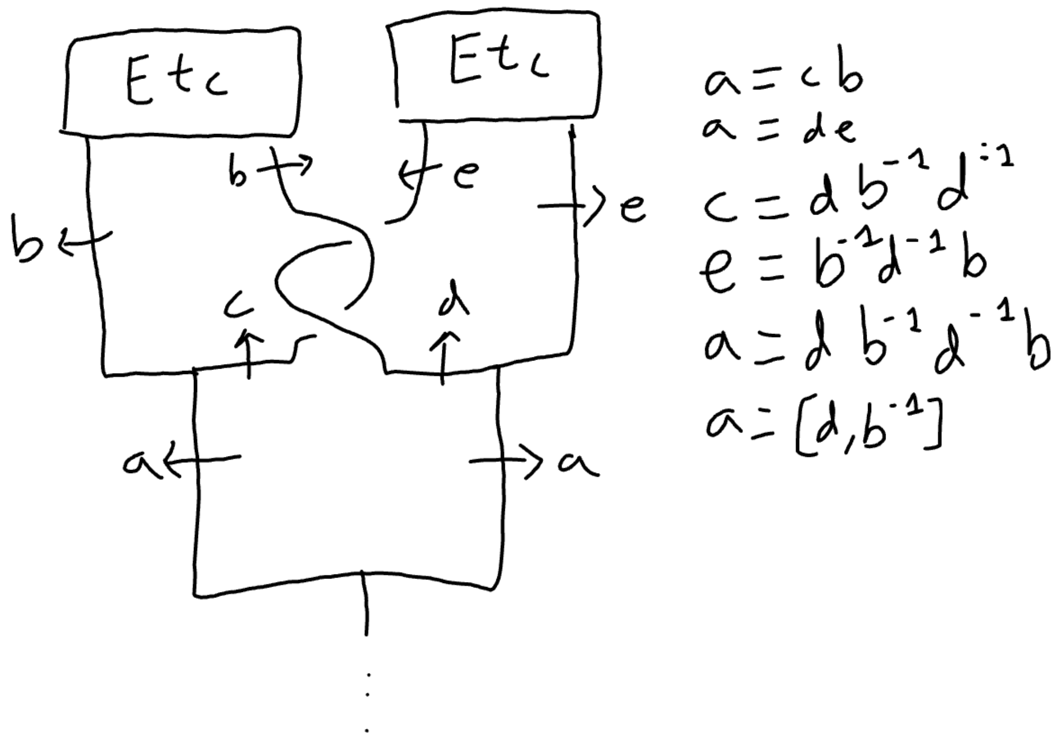


Figure 5.3: The Wirtinger process applied to one level of the AHB. There are no relations between b and d, so this is the free group of rank 2 ($\{b, d\} = F_2$), which makes sense since this image is homotopic to the complement of a punctured torus, which is known to have fundamental group F_2.

Then, by taking the direct limit, we can determine the full fundamental group of $S^3 - AHB$ to be as follows (see [BF50] for a rigorous proof): Let α be a finite sequence $\alpha = \alpha_1\alpha_2\ldots\alpha_l$ of length $l(\alpha) = l$ over the alphabet $a_i \in \{1, 2\}$. The string $\alpha 1 = \alpha_1\alpha_2\ldots\alpha_l 1$ and a similar construction exists for $\alpha 2$. Then the fundamental group is given by:

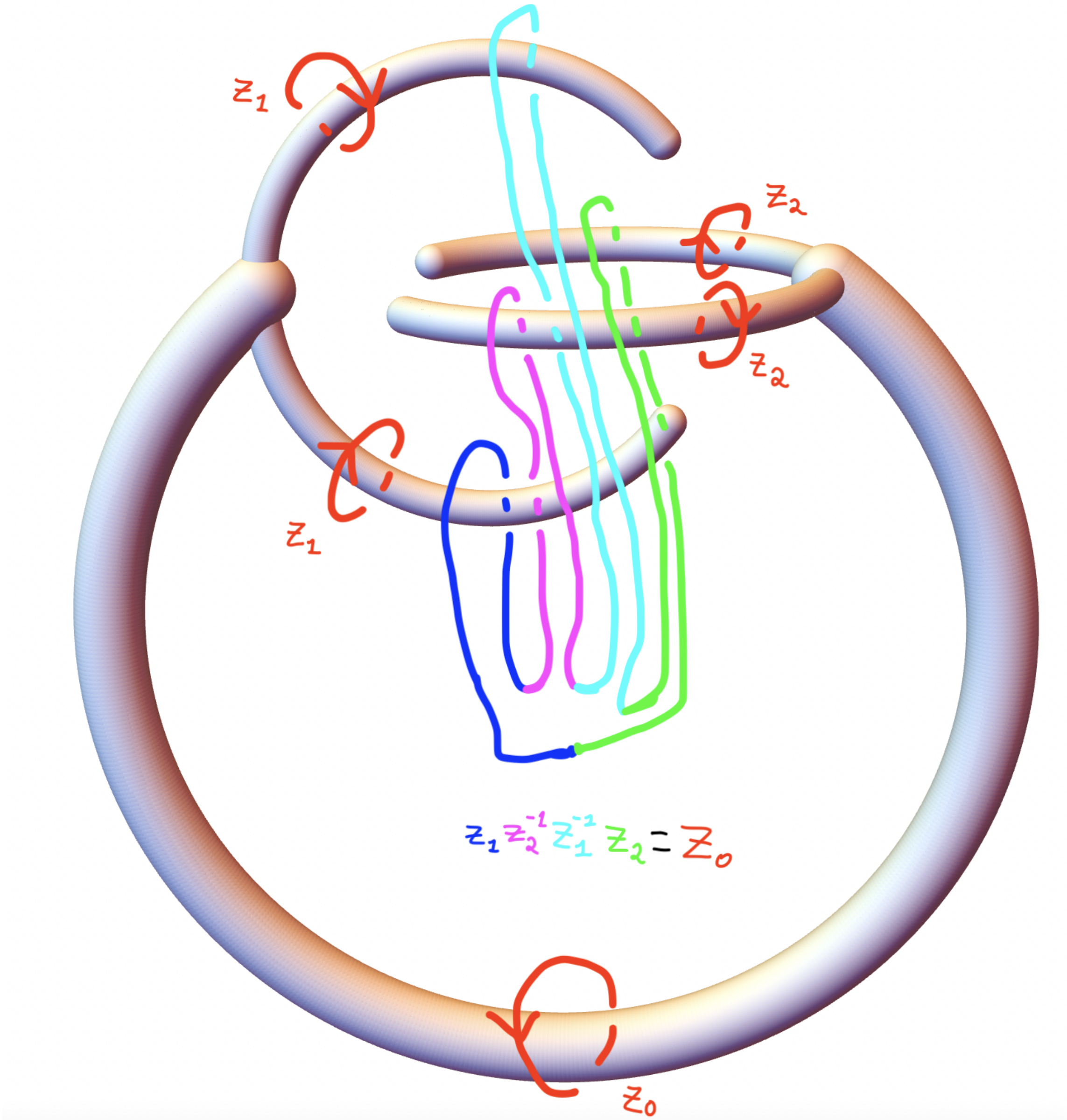
$$\pi_1(S^3 - AHB) = \{z_\alpha : l(\alpha) \geq 0,\ z_\alpha = [z_{\alpha 1}, z_{\alpha 2}^{-1}]\}$$

Where $[a, b] = aba^{-1}b^{-1}$ is the commutator. This group has several interesting properties: it is a locally free group, has infinite generators, and is a perfect group ($\pi_1 = [\pi_1, \pi_1]$).

Like all perfect groups, this group has trivial abelianization. This is exactly what we expect, since the first homology group $H_1(S^3 - AHB)$ is always the abelianization of the fundamental group, and in this case both are trivial.

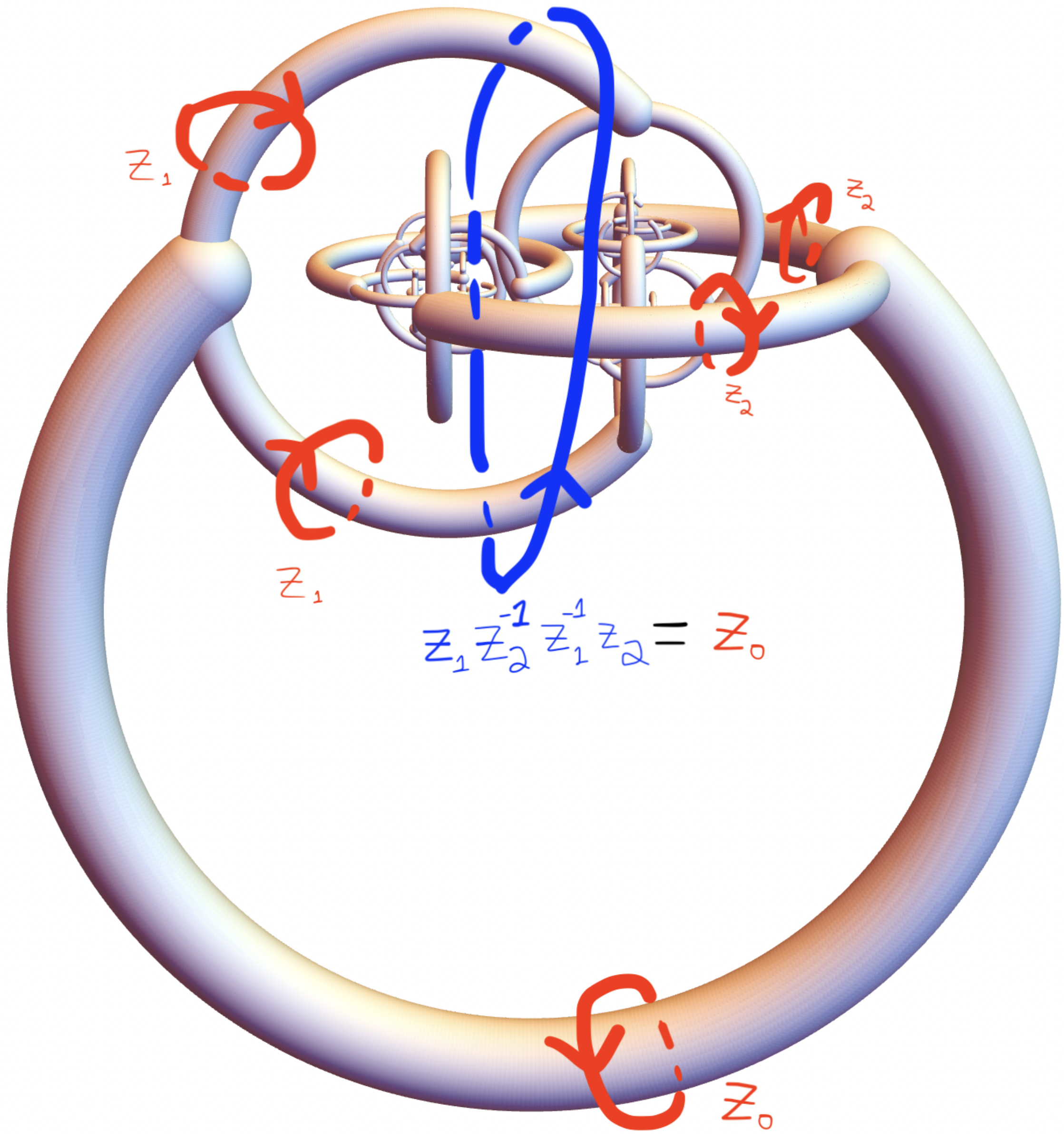
For a more intuitive understanding of what this fundamental group means, consider each α as selecting a specific handle of the Horned Sphere after going down a path and making left/right decisions at each decision point based on the string α. Then the relation $z_\alpha = [z_{\alpha 1}, z_{\alpha 2}^{-1}]$ becomes clear: if z_α is a grasping curve around a handle of the Horned Sphere, that curve can be visually verified as equivalent to the commutators of the grasping curves of that handles' children. See Figure 5.4 for a visual example showing the specific case of $z_0 = [z_1, z_2^{-1}]$, where z_0 has $\alpha = \emptyset$.

Now that we know the fundamental group and the homology groups of $S^3 - AHB$, can we figure out exactly what the complement actually is? A useful method, similar to how Hatcher derives the fundamental group, may be to construct the complement at each iteration of construction of Alexander's Horned Ball, then take the limit of those constructions.



(a) The commutator $[z_1, z_2^{-1}] = z_1 z_2^{-1} z_1^{-1} z_2$ shown on a visually simplified AHS. The rest of the recursive steps of the AHS have been removed so it is easier to see what is going on, but imagine that loops *cannot* be dragged over the end of the claps. See (b) for why the full AHS prevents that.

Figure 5.4: Continued on next page.



(b) The multicolored curve in (a) is homotopic to the blue curve in this image, which can be seen by mentally shrinking the sections of the curve that return to the basepoint until they are 'tight' against the hull of the AHS. This blue curve is then clearly homotopic to z_0. Unlike in (a), the rest of the AHS has been shown in this image.

Figure 5.4: Here we see a simple visual proof that $z_0 = [z_1, z_2^{-1}] = z_1 z_2^{-1} z_1^{-1} z_2$. The same logic applies at any depth, and since it is a recursive infinite structure, all elements are then the commutator of some other two elements.

Section 5.2

Connecting back to T^i

In a nice callback to our earlier chapter on the proof of Alexander Duality and related theorems, one theorem, T^i, states that any closed k-chain in the complement of a disc but bound some open $(k+1)$-chain in the complement.

Since a sphere with a hole in it is a disc, we can punch a hole in Alexander's Horned Sphere, which results in a topological disc. The grasping curve around the bottom of the resulting (which is exactly z_0 in Figure 5.4) is a closed 1-chain, so it must bound some 2-chain in the complement. What this 2-chain is can grant us some insight into the complements true nature. The 'punching of the hole' is not important to the result, but is instead only done so that we can directly apply T^i.

Following the 'instructions' laid out in the proof of T^i, we can figure out the shape of the bounded surface: take the half of the AHS to the left of the grasping curve, thicken it, and take the boundary of that. What you get is homotopic to a punctured torus, since thickening the wild points fill in all the small holes.

See Figure 5.5 for how this punctured torus appears on a standard embedding of the AHS, and Figure 5.6 for how this punctured torus appears on the upright, Cantor set-displaying embedding, which I find much easier to mentally manipulate.

Due to the recursive nature of Alexander's Horned Sphere, given any grasping curve z_α, we can find a punctured torus that covers all the children of the α-path. We can stretch out the boundary of these punctured tori so they overlap exactly with a generator of the previous generation's punctured torus, and union all these punctured tori together, taking the limit as $l(\alpha) \to \infty$, to create an 2-complex which successively

Figure 5.5: The punctured torus in the complement of the AHB which is bounded by the grasping curve z_0. Reprinted from [Wei] with red shading the author's own.

approximates a better and better exterior of the AHB.

Note that this object we've constructed is *not* a manifold, because where the curves where different 'generations' touch each other touch two punctured tori at the same time, which is not locally $\mathbb{R}^2$ and thus not a manifold.

Another notable property is that the punctured torus on the left and right side link each other: this is important because it means they are killing one of each others generators (the one that goes through the hole) if they were glued together. Therefore, since the child punctured torus kills a generator of the parent, and the pair punctured torus (left $\longleftrightarrow$ right) kills the other, all generators are killed, which would make the fundamental group of this object perfect - exactly what we would expect from the complement of the AHB.

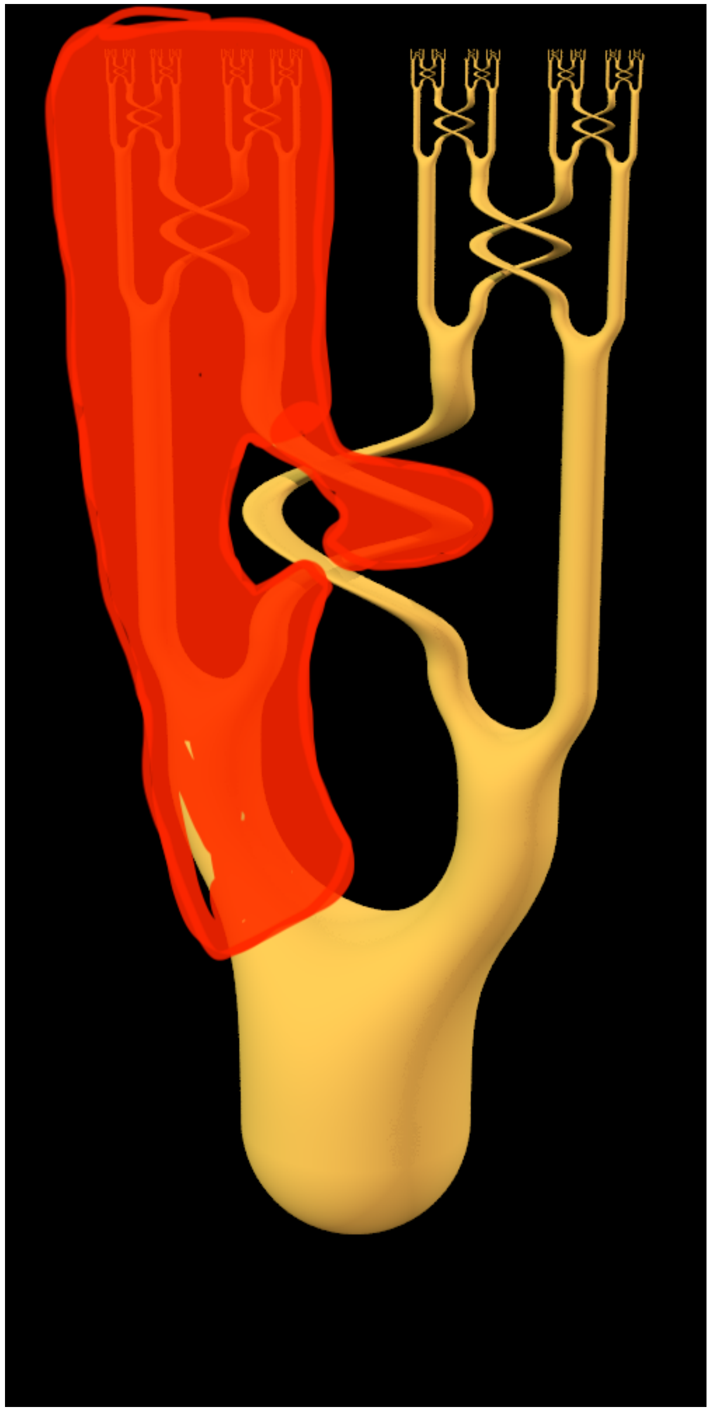


Figure 5.6: The punctured torus in the complement of the upright embedding of the AHB which is bounded by the grasping curve z_0. This diagram clearly shows how the AHS, its complement, and the punctured torus shown here both have rotational symmetry around a vertical axis. Original image from [Ché19], red drawing the author's own.

Section 5.3

An Introduction to Gropes

Luckily, people have discovered objects like this before: objects built through this recursively stitched-on punctured torus method are called *gropes*.

Definition 5.2. A *grope* is a 2-dimension complex with one boundary circle that is a union of surfaces. These surfaces are assumed to be compact oriented connected 2-manifolds with a single boundary circle. A new surface may only be adding to an existing one so that the boundary circle of the new surface is attached exactly to a generator of the fundamental group of the existing surface.

The *height* of a grope G is the length of the longest chain of 'stitched-on' surfaces. Therefore, a height 1 grope is just a surface, whereas a height 2 grope is shown in Figure 5.7, and a height 3 grope would be if any of the already stitched on surfaces in Figure 5.7 had another surface stitched on them.

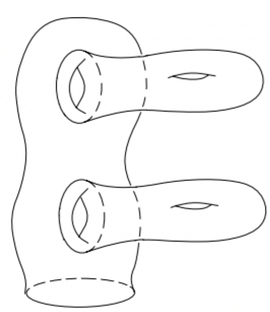


Figure 5.7: A grope of height 2. Note that the smaller punctured tori are each glued with their boundary going around a generators of the genus 2 base surface. Reprinted from [Tei04].

Note that gropes are not manifolds themselves, but have been described as being 'the next easiest thing after surfaces' because the singularities that arise are always in simple curves [Tei04].

In our case, due to the infinite nature of Alexander's Horned Sphere (the very thing that makes it wild), we will only be dealing with gropes of infinite height. However, finite height gropes can be used for many applications in knot theory and group theory, as in discussed in [Tei02].

Gropes are used in geometry group theory as the geometric analogues of commutators groups, which is exactly what we want, albeit with a rather complex commutator groups [Can78]. We will show that the complement of the AHB is a thickened grope - in other words, the grope is the spine of the crumpled cube.

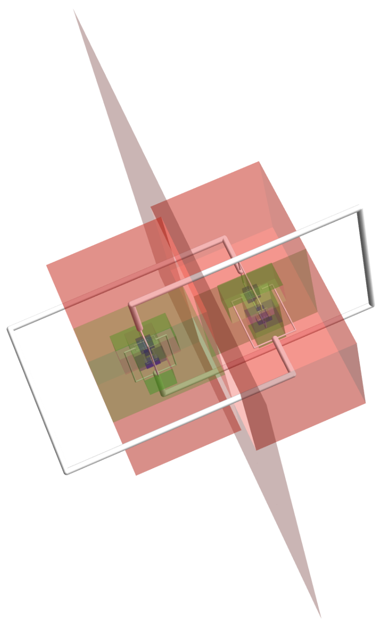
Section 5.4

Alexander's Grope

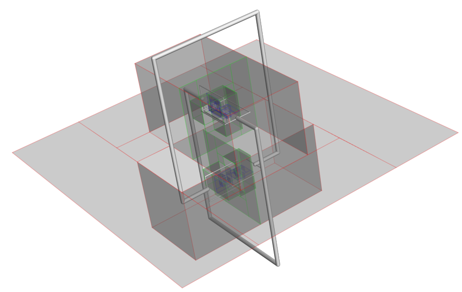
We can extrapolate from Figure 5.6 that the complement's boundary has two interlocking handles at each level of recursion. If we build such an object in Mathematica, as we did in Figure 5.8 we can clearly see how it fits in with a different embedding of the AHB.

Each of the two interlocking clasps of each layer of the AHB fit through each of the two interlocked handles of the grope (the handles shown in a rectangular form in Figure 5.8), so that with infinite regression, the grope will form a shell directly over all the clasps of the AHB. The images shown in the figure look the same at every level of regression, just scaled and rotated.

Therefore, by noting thing about the top level, we can note things about the entire object. For example, this grope has exactly the fundamental group we want it to: each disc portion is the commutator of its children handles' central disc portion; so



(a) The grope shown with successive depths of iteration in different colors (red, green, then blue) with the clasps of the AHB shown as the tubes.



(b) The grope shown in gray with successive depths of iteration having their edges in different colors (red, green, then blue) with the clasps of the AHB shown as the tubes.

Figure 5.8: The grope 'spine' of the crumpled cube, from two different perspectives, with an embedding of the AHB inside it to demonstrate how they perfectly fit together. Note that all open (white) space on the outside of the model includes the unshown the final connecting piece of the AHB that joins the two biggest clasps together.

in the diagram, the generator of the red plane is the commutator of the generators of the two green planes inside the red handles. This visual representation of this grope is a potentially novel creation done for this thesis.

Therefore we can form the full crumpled cube adding a 3-disc with this grope as its boundary. We have thus completely identified $S^3 - AHB$; we know its boundary, fundamental group, and visually can embed it.

Chapter 6

Variants of Alexander's Horned Sphere

In my studies of the Horned Sphere, we struggled to find an image that really represented the beauty of the mathematical object in an aesthetically pleasing way. One particular image online stood out as nice to my advisor, Professor Peter Doyle: Figure 6.1, generated by Professor Kathryn Lindsey of Boston College during her undergraduate years [Lin07].

Interestingly, Lindsey's rendering is *not* identical to the normal Alexander's Horned Sphere embedding (e.g. Figure 5.1). There is a link between the two halves that does not normally exist, which can be seen in the upper middle part of Figure 6.1. However, it is still a completely valid immersion of S^2 into $\mathbb{R}^3$, and since it is clear to see that, like the AHS, its exterior is not simply connected, Lindsey's Horned Sphere qualifies as a wild sphere.

We were curious about the properties of this Lindsey's Horned Sphere (LHS) variant of the AHS. Was the complement of Lindsey's Horned Ball (LHB) - obviously

Figure 6.1: Kathryn Lindsey's Horned Sphere (taken from [Lin07])

some kind of crumpled cube - the same as the complement of the AHB? If not, how did the differ? Are the fundamental groups of the complements of the LHB and AHB the same?

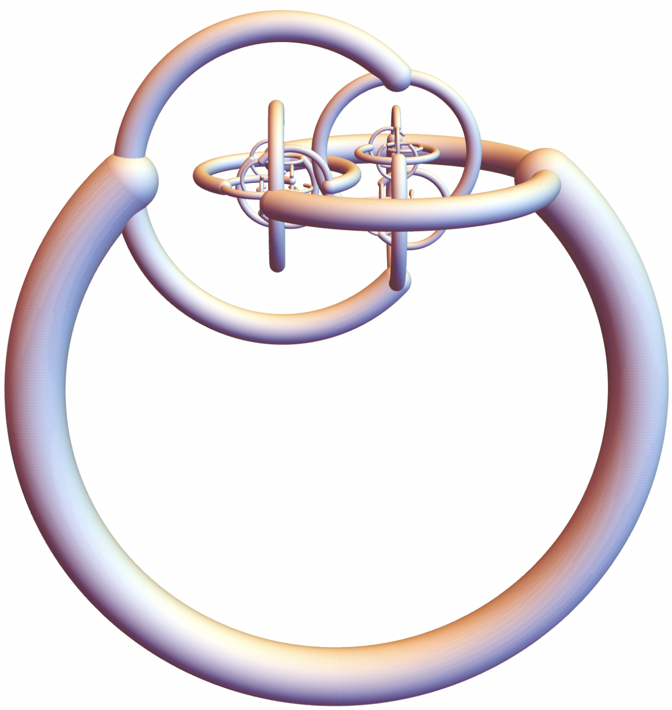
I began by remaking the LHS on my own, to see how the recursive generation process was different. The resulting image is Figure 6.2a. The only difference in generation is certain parameters (arc length, scaling between iterations), but this leads to that extra link occurring not just at the top level, but at every level.

Since the Cantor set-showing model of the AHS was so useful for understanding, I created a modified version of the model from [Ché19] for the LHS. In this image, Figure 6.2b, you can clearly see the extra link, which happens a generation after the two sides originally interacted.

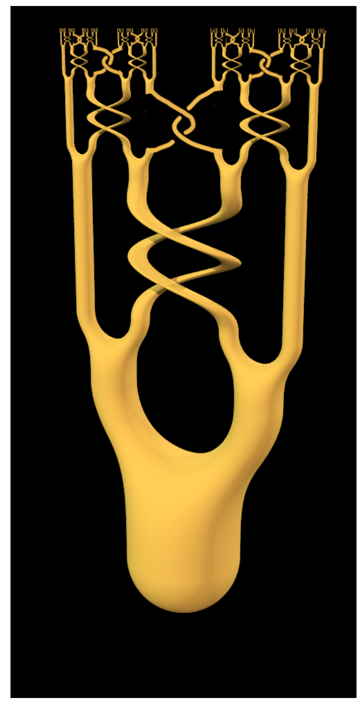
Finding a nice presentation of fundamental group is no longer such a easy process: our method in Figure 5.4 will no longer work since a curve around the two arms of the same handle (in other words, the two copies of z_1 in Figure 5.4) are no longer homotopic when placed on the LHS. This can easily be seen on the upright embedding as well. Nonetheless, we know the fundamental group must still be perfect, because its abelianization is trivial (by Alexander Duality), and thus all elements are commutators or products of commutators (all generators are commutators).

We can do the same process as we did with the AHS and apply T^i to find a surface bounded by the grasping curve around the bottom of the LHS. While for the AHS we got a punctured torus, for the LHS we get a *genus 2* punctured torus. See Figure 6.3 for a visual representation of this surface.

Examining deeper levels of the LHS, we can see that grasping curves around an arm of the LHS at any depth also bound genus 2 punctured tori.



(a) My recreation of the LHS using Mathematica.



(b) An upright embedding of the LHS, which shows the Cantor set at the top and the extra link that exists at each stage. Modified by the author from [Ché19].

Figure 6.2: Two models of the LHS

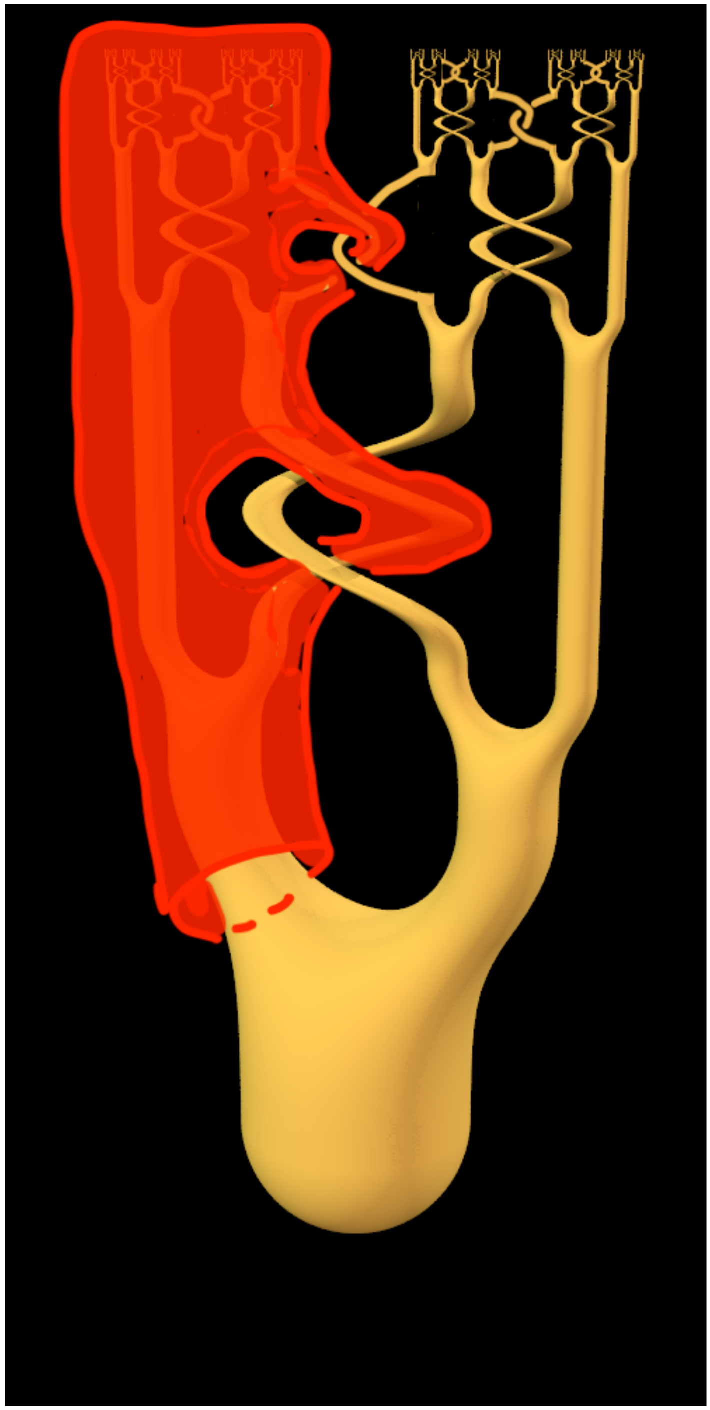
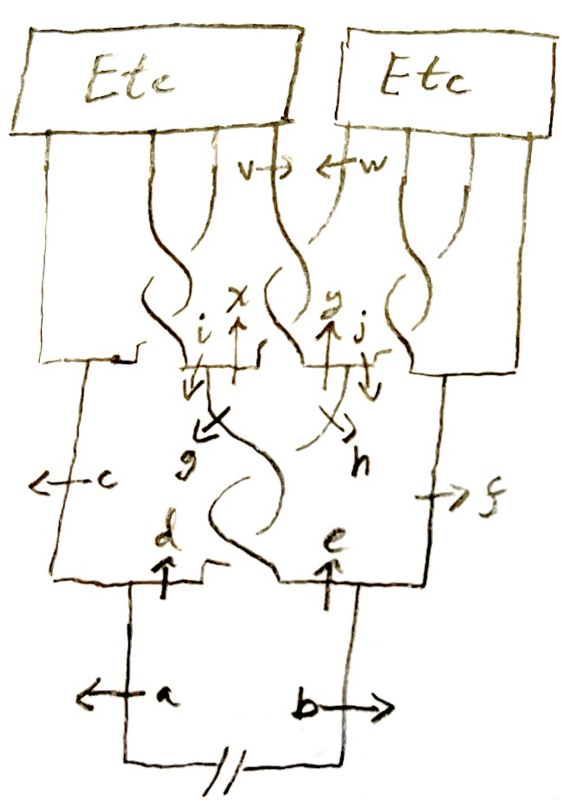


Figure 6.3: The genus 2 punctured torus that is bounded by the grasping curve around the bottom of the LHS, shown on the upright model of the LHS. Modified from [Ché19].

The perfect nature of the fundamental group, and levels of punctured tori seem to suggest a grope is at play here: the complement is likely homotopic to a tapered grope. However, gropes are very hard to visualize, and unfortunately we were not able to see the exact nature of the grope or create a diagram of it.

We can again attempt the Wirtinger process. See Figure 6.4 to see what results. Unfortunately, we were not able to find a nice presentation resulting from this. The LHS provides extra challenges in that there is non perfect symmetry: there are some branches with more links than other, which makes it hard to find a starting point for a knot diagram like this since there will likely be interactions with non-pictured parts (hence why $a \neq b$). However, if you add the relations $c = g^{-1}$ (which implies the equivalent relations $h = f^{-1}, x = v^{-1}, y = w^{-1}$), then the group of that first level collapses exactly to the fundamental group of the complement of the AHB: $F_2 = \{c, e\}$. Therefore, we can at least conclude that the fundamental group of $S^3 - LHB$ maps onto $\pi_1(S^3 - AHB)$.



$$a = dc$$
$$b = ef$$
$$d = ege^{-1}$$
$$e = g^{-1}hg$$
$$g = xi$$
$$h = yj$$
$$x = yv^{-1}y^{-1}$$
$$y = vw^{-1}v^{-1}$$

Figure 6.4: Wirtinger process applied to a level of the LHB. After significant simplification, I achieved the result $a = cf^{-1}vw^{-1}v^{-1}wc^{-1}f$, which is related to commutators. Also, note the relations implied by the diagram: $gc = hf = xv = yw$.

Chapter 7

Future Work

The primary avenue of future work is in finding and categorizing different variant types of the AHS, such as we have done with the LHS. For example, without the restrictions on having the embedding be built out of tori with proper arcs, any kind of knot can be placed in each generation, which ties into knot theory.

Similarly, like the LHS, we can add links between that do not happen until later generations - the LHS has an extra link between left and right sides one generation after their initial link, but that could instead be done after 2 generations, or at both the 1st and 2nd generation, or any possible combination. This would lead to increasingly more and more complex complements and wild spheres.

For example, while constructing a nice model of the AHS where the arms are titled 45 degrees relative to the previous generation, (see Figure 5.1), I made a version of the LHS that uses the same. Interestingly, there are now *two* links between the left and right halfs a generation after their original link; this can be seen in Figure 7.1 This would further complicate the structure of the complement.

A promising line of work would be to categorize, formalize, and further study all

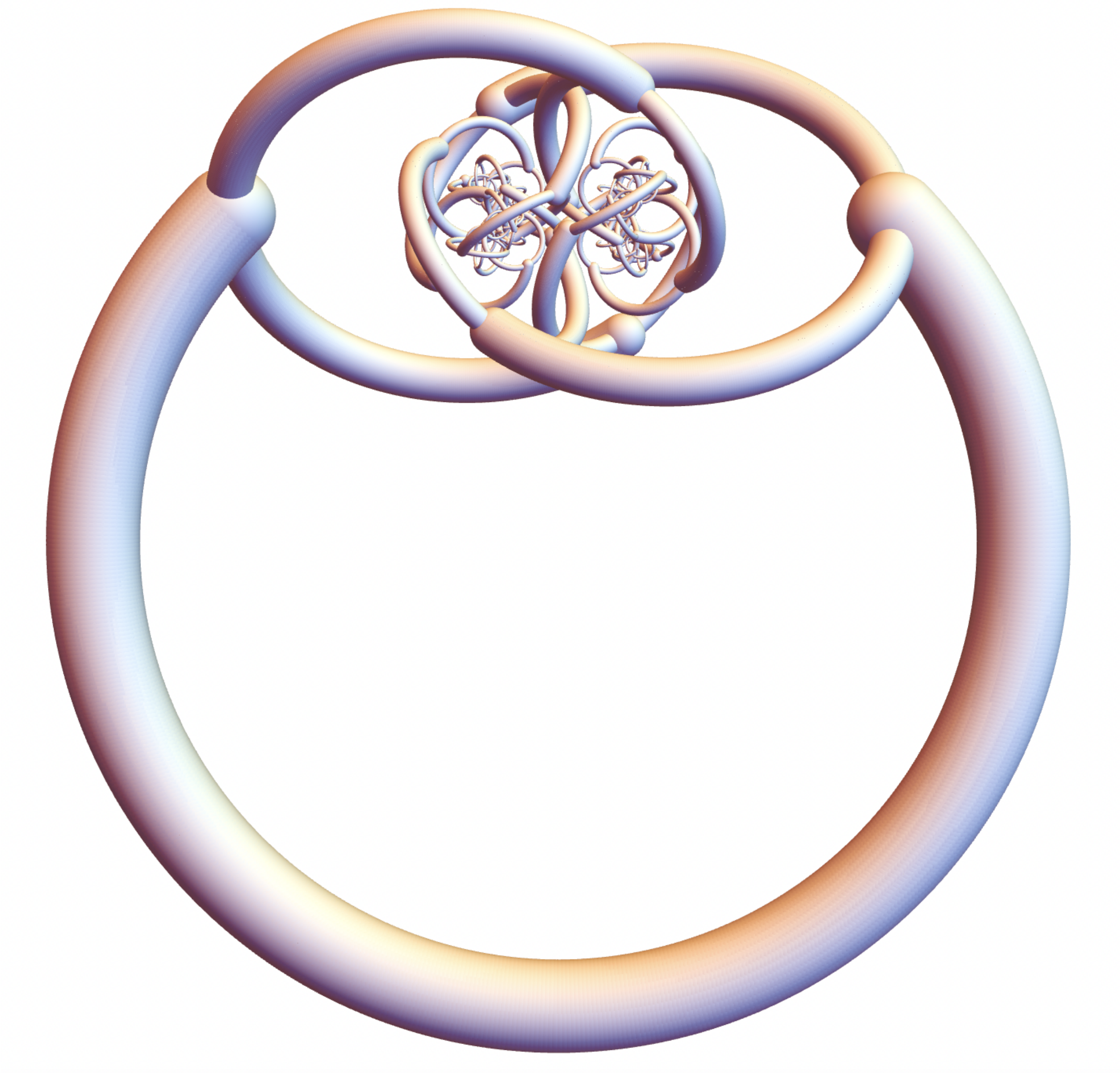


Figure 7.1: A variant horned sphere modified from the LHS where each generation has two links with its chiral pair from the previous generation, in addition to the normal link with its own chiral pair.

these variants of Alexander's Horned Sphere, which would certainly have ties to knot theory, grope theory, geometry group theory and more. I believe this would yield a deep relationship between wild spheres, gropes, and commutator groups, each of which have further applications: For example, in [AK13], they argue that wild surfaces such as these are instrumental to a particular perspective on quantum physics; Freedman and Quinn use gropes in [Fre90] to explore 4-dimensional manifolds.

James W. Cannon has done much excellent work on the topic of wild surfaces like these and it would be highly interesting to read through his full list of publications to understand our current bredth of understanding on the topic.

Finally, Mike Freedman raised the question at the crux of these variants: is any crumpled cube a tapered grope? I myself wonder what, if it is not the case, delineates the wild sphere of the crumpled cubes that are gropes against those that are not.

Appendix A

Chains

The specifics of chains sides are cited by Alexander from a paper by Veblen [Veb05], a quick summary will be made below.

We can determine the sides as follows: complete the lines to which the edges of the chain k belon and thus obtain a system of lines which subdivide each other into a finite number of line segments and rays $b_1, b_2, \ldots, b_n$, and subdivide the plane into a finite number of convex region $a_1, a_2, \ldots, a_m$. Now, the boundaries of each a_i are chains made of elements of b_j and their end points. We can form the expressions

$$a_i = b_{i_1} + \cdots + b_{i_k} \qquad (i = 1, 2, \ldots m)$$

to designate the boundary chain of each region a_i. The expressions for any a_i, a_j can be combined by adding corresponding members and reducing the coefficients modulo 2.

It can be shown that any chain such as k composed of elements b_i and their endpoints can be derived from the elementary chains (defining each a_i) in two and only two possible ways. For example, consider a simple case, let k be the boundary of a simple shape, such as a regular polygon. For simplicity, I will use an equilateral triangle, but any will suffice. Extend each side to a line. Label the exterior regions, A, B, C, and the internal region X. Then label each line segment and ray, so that none intersect at more than a finite number of points. As described above, we can form expressions for each of A, B, C, X out of sums of these line segments. The chain k can clearly be made in exactly two ways (up to modulo two coefficients of each member) out of the regions we have defined. $k = X$ or $k = A + B + C$. These two options define our two sides. If a point is inside X, it is on one side, let's call 'inside' of k. If the point is instead in A, B, C, it is on the other side, 'outside'.

This construction argument can be applied to any chain, even if it is not a regular polygon, since with a chain there will always be a finite number of rays and line segments generated by extending all segments of the chain to lines. More generally, for any region a_i formed by the extended lines of the chain, the boundary of each a_i occurs in exactly one of those two ways. So the points of the plane fall into two classes according to if they belong in the interior or boundary of a region in the first combination or the second. These two classes are what we will refer to as sides. The labels of interior and exterior are arbitrary and, unlike the two classes of side, are not well-defined, since both regions may contain points at infinity due to the use of rays so neither can be specified as exterior. Therefore, we simply call them the two distinct sides.

Appendix B

Mathematica Code to Generate Horned Spheres and Related Objects

The following code was used to generate the initial AHS model, and was tweaked to create the LHS, other variants such as Figure 7.1, and in combination with a model of each stage of the grope to create Figure 5.8.

In order to create the LHS, change the 'endpts' variable to 2.40 and the thickness of the tube to 0.1 (the line **a=Tube[toruspts,0.1];**). The variant such as Figure 7.1 was created by changing the rotation transforms to be 45 degrees instead of 90.

```
(*Get points up to a chosen level along a torus*)
quality = 500; (*Number of points along each arc of the torus*)
endpts = 2.47; (*Angle of each side of arc of torus*)
toruscurve[u_]:={−Cos[u], 0, Sin[u]};
toruspts = Table[{1, 0, 0} + toruscurve[i ∗ endpts/quality], {i, −quality, quality, 2}];
```

```
a = Tube[toruspts, 0.08];

RegressR[a_]:=GeometricTransformation[a, Composition@@{
TranslationTransform[toruscurve[endpts] + {1, 0, 0}], (*Takes to target pt*)
ScalingTransform[{0.5, 0.5, 0.5}, {0, 0, 0}], (*Scales by scale factor*)
RotationTransform[{{0, 0, 1}, {1, 0, 0}}](*Rotates to new angle*)
}];
RegressL[a_]:=GeometricTransformation[a, Composition@@{
TranslationTransform[toruscurve[-endpts] + {1, 0, 0}], (*Takes to target pt*)
ScalingTransform[{0.5, 0.5, 0.5}, {0, 0, 0}], (*Scales by scale factor*)
RotationTransform[{{0, 0, 1}, {0, 1, 0}}], (*Rotates to new angle*)
RotationTransform[{{0, 1, 0}, {1, 0, 0}}]
}];
Regress[a_, depth_]:=If[depth==1,
(*Base Case*)
{a}
,
(*False(iterative)Case*)
Join[{a}, Regress[RegressR[a], depth - 1], Regress[RegressL[a], depth - 1]]
];
Graphics3D[Regress[a, 5], Boxed->False]
```

Bibliography

[AK13] Torsten Asselmeyer-Maluga and Jerzy Król. "Quantum Geometry And Wild Embeddings As Quantum States". In: *International Journal of Geometric Methods in Modern Physics* 10.10 (Oct. 2013). DOI: 10.1142/s0219887813500552.

[Ale20] J. W. Alexander. "A Proof of Jordan's Theorem About a Simple Closed Curve". In: *Annals of Mathematics* 21.3 (1920), pp. 180–184. ISSN: 0003486X. URL: http://www.jstor.org/stable/2007256.

[Ale22] J. W. Alexander. "A Proof and Extension of the Jordan-Brouwer Separation Theorem". In: *Transactions of the American Mathematical Society* 23.4 (1922), pp. 333–349. ISSN: 00029947. URL: http://www.jstor.org/stable/1988883.

[BF50] W. A. Blankinship and R. H. Fox. "Remarks on Certain Pathological Open Subsets of 3-Space and Their Fundamental Groups". In: *Proceedings of the American Mathematical Society* 1.5 (1950), pp. 618–624. ISSN: 00029939, 10886826. URL: http://www.jstor.org/stable/2032290.

[Can78] James W. Cannon. "The recognition problem: What is a topological manifold?" In: *Bulletin of the American Mathematical Society* 84 (1978), pp. 832–866.

[Ché19] Arnaud Chéritat. *Alexander's Horned Sphere*. Sept. 2019. URL: https://im.icerm.brown.edu/portfolio/alexanders-horned-sphere/.

[Fre90] Michael H. Freedman. *Topology of 4-manifolds*. eng. Princeton mathematical series, 39. Princeton, N.J: Princeton University Press, 1990. ISBN: 0691085773.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge: Cambridge University Press, 2002, pp. xii+544.

[Lin07] Kathryn Lindsey. *A variant of the Alexander horned sphere*. 2007. URL: https://www.kathrynlindsey.com/images/repsci200706.jpg.

[Tei02] Peter Teichner. "Knots, von Neumann signatures, and grope cobordism". In: *Proceedings of the International Congress of Mathematicians* 2 (2002), pp. 437–446.

[Tei04] Peter Teichner. "What is a Grope?" In: *Notices of the AMS* 51.8 (Sept. 2004), pp. 894–895. URL: http://http://www.ams.org/notices/200408/what-is-teichner.pdf.

[Veb05] Oswald Veblen. "Theory on Plane Curves in Non-Metrical Analysis Situs". In: *Transactions of the American Mathematical Society* 6.1 (1905), pp. 83–98. ISSN: 00029947. URL: http://www.jstor.org/stable/1986378.

[Wei] Eric W. Weisstein. *Alexander's Horned Sphere. From MathWorld—A Wolfram Web Resource*. URL: https://mathworld.wolfram.com/AlexandersHornedSphere.html.