Sheaf Cohomology

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DRP

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Philosophy

- (local-global) gluing together "local" things to get a "global" thing
  - if something is true locally, is it true globally?
- (Yoneda) understand a space by studying the functions on it

Sheaves are a tool at the intersection of these two philosophies.

Presheaves

A **presheaf** F on a topological space X is the following data:

- for each open subset $U \subset X$, it assigns an abelian group $F(U)$
- (restriction maps) for each inclusion of open subsets $U \subset V$, there is a group homomorphism $F(V) \to F(U)$

Sheaves

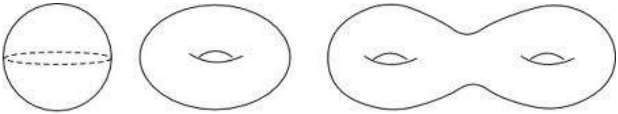
Let $U \subset X$ be an open subset, with an open covering $\mathcal{U}$.

A **sheaf** $\mathscr{F}$ is a presheaf such that

1. (locality) if $s, t \in \mathscr{F}(U)$ are such that $s|_{U_i} = t|_{U_i}$ for each $U_i \in \mathcal{U}$, then $s = t$.
2. (gluing) if for each $i \in I$ there is a section $s_i \in \mathscr{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all $i, j \in I$, then there exists a section $s \in \mathscr{F}(U)$ such that $s|_{U_i} = s_i$ for each $i \in I$.

Compact Riemann surfaces

- even topologically distinct compact Riemann surfaces have the exact same global holomorphic functions



Source: greatmathmoments

Riemann-Roch

Theorem. For a compact Riemann surface X,

$$\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X(D)) - \dim_{\mathbb{C}} H^1(X, \mathcal{O}_X(D)) = \deg(D) + 1 - g.$$

Philosophy

- To motivate sheaves, I want to talk about 2 central philosophies in math
  - the first is the relation between <u>local and global things.</u>
    - if I have a collection of small pieces that fit together, I want to be able to combine them to get a bigger thing
      - eg manifolds
    - a question I can also ask is whether something being true "locally", if it is true "globally"
      - color of individual pixels
      - p-adic numbers
- (<u>Yoneda</u>) We can learn a lot about ~~the~~ a space by studying the functions on it.
  - like the relation of an object to its environment

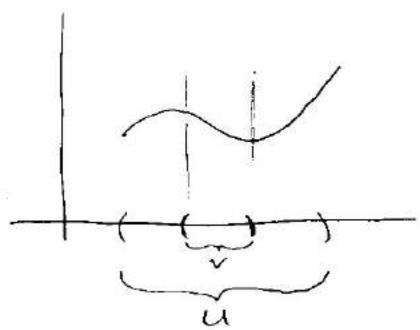
Presheaves

~~Everything~~ I'm ~~going to~~ discuss about

- Sheaves and presheaves can be defined quite generally, but this def for our purposes.

$$V \hookrightarrow U$$
$$F(V) \hookleftarrow F(U)$$

- ~~The reason for~~ this "contravariance" is actually a natural thing.
- Sheaves really characterize how functions behave, so let's consider continuous functions ~~from~~ on $\mathbb{R}$ (to $\mathbb{R}$)



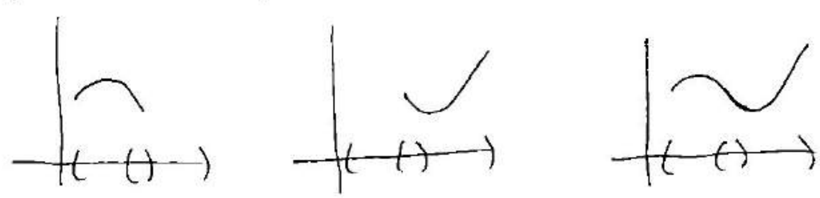
- As we can see, every cts function on U naturally "restricts" to one on V.

#4) Sheaves

- Sheaves are presheaves w/ 2 more properties that relate them to the local-global philosophy from earlier.

  1. (locality) If things look the same everywhere I zoom in, they are the same thing overall.
     - It's unintuitive why this needs to be said, but we'll cover it when we get to an abstract example.

  2. (gluing) If small pieces agree on their overlap, I should be able to glue them together to get a bigger pieces



HW 7 pt ex

• • Presheaf: assign to every open set the group $\mathbb{Z}$. Restr. maps are id.

$\emptyset \to \mathbb{Z}$
$\{p\} \to \mathbb{Z}$
$\{q\} \to \mathbb{Z}$
$\{p,q\} \to \mathbb{Z}$

$$\begin{tikzcd} & \emptyset \arrow[dl, hook] \arrow[dr, hook] \arrow[dd] & \\ \{p\} \arrow[dr, hook] & & \{q\} \arrow[dl, hook] \\ & \{p,q\} & \end{tikzcd} \qquad \begin{tikzcd} & \mathbb{Z} & \\ \mathbb{Z} \arrow[ur] & & \mathbb{Z} \arrow[ul] \\ & \mathbb{Z} \arrow[ul] \arrow[ur] \arrow[uu] & \end{tikzcd}$$

Q: Is this a sheaf?

1. (locality) The empty set is covered by the empty set...

   $F(\emptyset) = \mathbb{Z}$, so locality says if $m, n \in \mathbb{Z}$ agree on each set in this cover, they are the ~~q~~ same.

   - But this is auto true - but $m \neq n$ in general.

- This pretty much tells us the only thing we can assign to $\emptyset$ is 0.

$$\begin{tikzcd} & 0 & \\ \mathbb{Z} \arrow[ur, "0"] & & \mathbb{Z} \arrow[ul, "0"'] \\ & \mathbb{Z} \arrow[ul, "id"] \arrow[ur, "id"'] \arrow[uu, "0"] & \end{tikzcd}$$

- Now is it a sheaf?

2. (gluing) $\{p,q\}$ covered by $\{p\}, \{q\}$.
   - let $m \in F(p)$, $n \in F(q)$, $m \neq n$
   - $p \cap q = \emptyset$, m, n trivially agree over $\emptyset$,
   $\Rightarrow \exists\, x \in F(p,q) = \mathbb{Z}$ st $x|_p = m$, $x|_q = n$
   - Impossible if $p \neq q$ and $x \in \mathbb{Z}$!

Need $F(p,q) = \mathbb{Z} \oplus \mathbb{Z}$.

#11 Compact C-mflds

- Let's return to the question of if a local thing "comes from" a global thing.
- In $\mathbb{R}$, this is pretty clearly true:

extend w/ lines

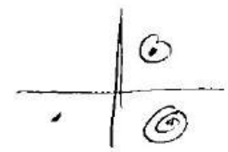
True for smooth too...

- An important class of spaces are complex manifolds, particularly (compact) Riemann surfaces.
  - e.g. arise as space of complex solutions to alg. eqs in 2 vars.
  - often "compactify" by "adding pts at ∞"
- eg Sheaf of holo fts.
- What are the global ~~hol~~ sections?
  - compact $\Rightarrow$ ftn is bounded
  - Louville $\Rightarrow$ ftn is constant
- Problem here - I can have a locally nonconstant ftn, and this says it doesn't come from a global one!
- Related problem: I can't differentiate compact RSs just by looking @ their global sections!

Need sheaves

#1#1 Exactness

- Let me show you how to phrase the "failure of local sections to come from global ones"
— more algebraically.
- In my project, we looked at mero fts
- Specify some poles D:



- $\mathcal{O}_X(D)$ - sheaf of mero fts w/ "at most" those poles
- holo fts are clearly in there (no poles) $\mathcal{O}_X \hookrightarrow \mathcal{O}_X(D)$
- Quotient to get non holo mero fts $\mathcal{H}_D$:

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{H}_D \longrightarrow 0$$

- As q:
- No. If it were exact, the theory of sheaves would tell me that given some Laurent expansions around some poles, I can find a globally mero ftn
...

) eg when its false. $\mathbb{C}$-log

- I can locally define this for all pts in $\mathbb{C}$, not taking any-thing crossing negative real for example.
- But no global log...

Cohomology

- In our $\mathbb{C}X$,

$$0 \to \mathcal{O}_X(X) \to \mathcal{O}_X(D)(X) \to \mathcal{H}_D(X) \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X(D)) \to .$$

RR

- A major theorem tells us how to compute $H^1(X, \mathcal{O}_X(D))$
- The surprising thing here is the g - its a geometric property - the # of "holes" of the surface
- deg is "counting" the number of poles + multiplicity/order