Representations of Cyclic Groups

from *Representations and Characters of Groups*, 2nd ed., by Gordon James & Martin Liebeck

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Definitions

Definitions: A group consists of a set G, together with a rule for combining two elements $g, h \in G$ to form another element, $gh \in G$. This rule is called the group operation. The group operation must satisfy the following axioms:

1. $\forall\, g, h, k \in G$, $(gh)k = g(hk)$;
2. $\exists\, e \in G$ s.t. $\forall\, g \in G$, $eg = ge = g$;
3. $\forall\, g \in G$, $\exists\, g^{-1} \in G$ s.t. $gg^{-1} = g^{-1}g = e$.

Homomorphisms and Isomorphisms

Definitions: Given groups G and H, a homomorphism from G to H is a function $\theta : G \to H$ which satisfies $(g_1 g_2)\theta = (g_1\theta)(g_2\theta) \; \forall \; g_1, g_2 \in G$.

General Linear Groups

Definition: The general linear group of degree n over F is the set of invertible nxn matrices with entries in a field F, together with the group operation of matrix multiplication. We denote this group $\mathrm{GL}(n, F)$.

Representations

Definitions: A representation of a group G over a field F is a homomorphism $\rho : G \to \mathrm{GL}(n, F)$, for some $n > 0 \in \mathbb{Z}$. n is called the dimension of ρ.

Definition: A group G is generated by an element $a \in G$ if $\forall\ g \in G$, $g = a^n$ for some $n \in \mathbb{Z}$. We denote $G := \langle a \rangle$, and say that G is the cyclic group generated by a.

Definitions: If $a^n = 1$ for some $n \geq 1 \in \mathbb{Z}$, then $\langle a \rangle$ is finite. If $r \in \mathbb{Z}$ is the least possible integer ≥ 1 s.t. $a^r = 1$, then $\langle a \rangle = \{1, a, ..., a^{r-1}\}$, and r is the order of $\langle a \rangle$. We denote the cyclic group of order n as C_n.

Examples

1-dimensional representations of C_4

We begin by examining the 1-dimensional representations of $C_4 := \langle a \mid a^4 = 1 \rangle$.

Let one such representation be denoted $\rho : C_4 \to \mathrm{GL}(1, \mathbb{C})$. Recall that a representation is a homomorphism by definition. Thus, since $a^4 = 1$, we must also have $(a\rho)^4 = 1\rho = [1]$. Let $a\rho = [\alpha]$. Since $[\alpha]$ is a 1x1 matrix, we have $[\alpha]^4 = [\alpha^4]$.

Then $[\alpha^4] = [\alpha]^4 = (a\rho)^4 = 1\rho = [1]$, so $\alpha^4 = 1$.

1-dimensional representations of C_4

There are four such numbers $\in \mathbb{C}$: 1, -1, i, $-i$.

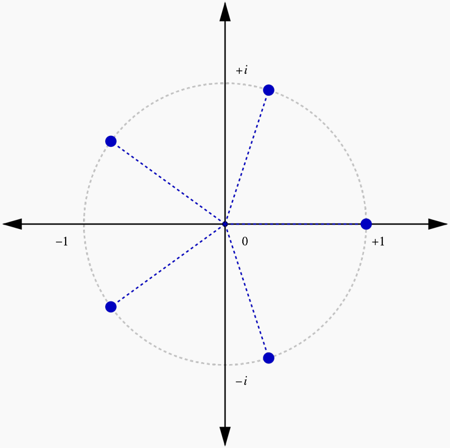
So we have four distinct 1-dimensional representations of C_4:

- $\rho_1 : C_4 \to \mathrm{GL}(1, \mathbb{C}); a \mapsto [1]$
- $\rho_2 : C_4 \to \mathrm{GL}(1, \mathbb{C}); a \mapsto [-1]$
- $\rho_3 : C_4 \to \mathrm{GL}(1, \mathbb{C}); a \mapsto [i]$
- $\rho_4 : C_4 \to \mathrm{GL}(1, \mathbb{C}); a \mapsto [-i]$

Since ρ_1 sends every element of C_4 to [1], the identity, we call ρ_1 the trivial representation.

Roots of Unity

Definition: An <u>nth root of unity</u> is a number $z \in \mathbb{C}$ such that $z^n = 1$. In general, nth roots of unity are given by $e^{\frac{2\pi ik}{n}}$ for $0 \le k < n$. Every nth root of unity is a power of $\omega_n = e^{\frac{2\pi i}{n}}$.



1-dimensional representations of C_3

We similarly delineate the 1-dimensional representations of $C_3 := \langle a \mid a^3 = 1 \rangle$. Any representation $\rho : C_3 \to \mathrm{GL}(1, \mathbb{C})$ must take a to a 1x1 matrix with a 3rd root of unity as its sole entry.

The 3rd roots of unity are: 1, $e^{\frac{2\pi i}{3}}$, $e^{\frac{4\pi i}{3}}$.

So we have three distinct 1-dimensional representations of C_3:

- $\rho_1 : C_3 \to \mathrm{GL}(1, \mathbb{C})$; $a \mapsto [1]$
- $\rho_2 : C_3 \to \mathrm{GL}(1, \mathbb{C})$; $a \mapsto [e^{\frac{2\pi i}{3}}]$
- $\rho_3 : C_3 \to \mathrm{GL}(1, \mathbb{C})$; $a \mapsto [e^{\frac{4\pi i}{3}}]$

Again, ρ_1 is the trivial representation, as it sends every element of C_3 to the identity.

2-dimensional representations of C_4

We may now examine an example of a 2-dimensional representation of C_4.

Let $\rho : C_4 \to \mathrm{GL}(2, \mathbb{C})$ such that $a\rho = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}$.

Then $(a\rho)^4 = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1\rho$, so ρ is indeed a representation.

2-dimensional representations of C_4

Here is another example of a 2-dimensional representation of C_4.

Let $\rho : C_4 \to \mathrm{GL}(2, \mathbb{C})$ such that $a\rho = \begin{bmatrix} -i-2 & i+1 \\ -2i-2 & 2i+1 \end{bmatrix}$. Call this matrix A.

Then $(a\rho)^4 = A^4 = \begin{bmatrix} -i-2 & i+1 \\ -2i-2 & 2i+1 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1\rho$, so ρ is indeed a representation.

2-dimensional representations of C_4

We see also that our matrix A is diagonalizable.

Let $T = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$.

Then $T^{-1}AT = \begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} -i-2 & i+1 \\ -2i-2 & 2i+1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$.

So $\begin{bmatrix} -i-2 & i+1 \\ -2i-2 & 2i+1 \end{bmatrix}$ and $\begin{bmatrix} i & 0 \\ 0 & -1 \end{bmatrix}$ are similar matrices.

Conclusion

Conclusion

The 1-dimensional representations described earlier are enough to construct every representation of a cyclic group.

Beyond Cyclic Groups

Definition: A group G is abelian if $g_1 g_2 = g_2 g_1 \; \forall \; g_1, g_2 \in G$.

Our result with regards to cyclic groups can be extended to all finite abelian groups; there are finitely many 1-dimensional representations of finite abelian groups, and they can be used to construct every representation of the group.

For finite groups in general, not necessarily abelian, we still have finitely many irreducible representations, though these may not be 1-dimensional.

Credits

Bibliography

James, Gordon, and Martin Liebeck. *Representations and Characters of Groups*. 2nd ed., Cambridge University Press, 2001.

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