A taste of Intuitionistic Logic

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To φ or not to φ

Law of excluded middle

$\varphi \vee \neg\varphi$

Natural Deduction

Let **PV** be an infinite set of *propositional variables*.

Definition

Let Δ be the least set such that:

- $\bot \in \Delta$
- $\mathrm{PV} \subset \Delta$
- $\varphi, \psi \in \Delta$ then $(\varphi \wedge \psi), (\varphi \vee \psi), (\varphi \rightarrow \psi) \in \Delta$

Δ is our set of formulas.

Natural Deduction

Definition

A **judgment** is a pair consisting of a finite set of formulas Γ and a formula φ, and we denote it by $\Gamma \vdash \varphi$.

Natural Deduction

Classical Propositional Calculus

$$\Gamma, \varphi \vdash \varphi \;(\mathrm{Ax}) \qquad \Gamma \vdash \varphi \vee \neg \varphi \;(\mathrm{Ax})$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\;(\to\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\;(\to\mathrm{E})$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\;(\wedge\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\;(\wedge\mathrm{E})\;\frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\;(\vee\mathrm{I})\;\frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \vartheta \quad \Gamma, \psi \vdash \vartheta \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \vartheta}\;(\vee\mathrm{E})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\;(\bot\mathrm{E})$$

Natural Deduction

Definition

We inductively define a **derivable judgment** as any judgment that is either an axiom or is derived from the rules of inference.

Definition

A **theorem** is a derivable judgment with $\Gamma = \varnothing$.

Natural Deduction

$$
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  \dfrac{}{\{\neg\neg p, p\} \vdash p}(\mathrm{Ax})
  \quad
  \dfrac{
    \dfrac{
      \dfrac{}{\{\neg\neg p, \neg p\} \vdash \neg\neg p}(\mathrm{Ax})
      \quad
      \dfrac{}{\{\neg\neg p, \neg p\} \vdash \neg p}(\mathrm{Ax})
    }{\{\neg\neg p, \neg p\} \vdash \bot}(\rightarrow E)
  }{\{\neg\neg p, \neg p\} \vdash p}(\bot E)
  \quad
  \dfrac{}{\{\neg\neg p\} \vdash p \vee \neg p}(\mathrm{Ax})
}{
  \dfrac{\{\neg\neg p\} \vdash p}{\{\} \vdash \neg\neg p \rightarrow p}(\rightarrow I)
}(\vee E)
$$

Natural Deduction

Classical Propositional Calculus

$$\Gamma, \varphi \vdash \varphi \; (\mathrm{Ax}) \qquad \Gamma \vdash \varphi \vee \neg\varphi \; (\mathrm{Ax})$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \; (\to\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \; (\to\mathrm{E})$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \; (\wedge\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \; (\wedge\mathrm{E}) \; \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \; (\vee\mathrm{I}) \; \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \vartheta \quad \Gamma, \psi \vdash \vartheta \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \vartheta} \; (\vee\mathrm{E})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \; (\bot\mathrm{E})$$

Natural Deduction

Intuitionistic Propositional Calculus

$$\Gamma, \varphi \vdash \varphi \ (\mathrm{Ax})$$

~~$\Gamma \vdash \varphi \vee \neg \varphi$ (Ax)~~

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi} \ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi} \ (\to\mathrm{E})$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi} \ (\wedge\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi} \ (\wedge\mathrm{E}) \ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi} \ (\vee\mathrm{I}) \ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \vartheta \quad \Gamma, \psi \vdash \vartheta \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \vartheta} \ (\vee\mathrm{E})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi} \ (\bot\mathrm{E})$$

Natural deduction

Intuitionistic Propositional Calculus

$$\Gamma, \varphi \vdash \varphi \ (\mathrm{Ax})$$

$$\frac{\Gamma, \varphi \vdash \psi}{\Gamma \vdash \varphi \to \psi}\ (\to\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \to \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\ (\to\mathrm{E})$$

$$\frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\ (\wedge\mathrm{I}) \qquad \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \varphi}\ (\wedge\mathrm{E})\ \frac{\Gamma \vdash \varphi \wedge \psi}{\Gamma \vdash \psi}$$

$$\frac{\Gamma \vdash \varphi}{\Gamma \vdash \varphi \vee \psi}\ (\vee\mathrm{I})\ \frac{\Gamma \vdash \psi}{\Gamma \vdash \varphi \vee \psi} \qquad \frac{\Gamma, \varphi \vdash \vartheta \quad \Gamma, \psi \vdash \vartheta \quad \Gamma \vdash \varphi \vee \psi}{\Gamma \vdash \vartheta}\ (\vee\mathrm{E})$$

$$\frac{\Gamma \vdash \bot}{\Gamma \vdash \varphi}\ (\bot\mathrm{E})$$

From Classical to Intuitionistic

What is the difference?

Semantics of Classical Propositional Calculus

$\neg(p \wedge q) \rightarrow \neg p \vee \neg q$

Semantics of Classical Propositional Calculus

Definition

A **classical valuation** is a function from **PV** to $\{0, 1\}$.

Definition

Given a valuation v, we define the **value function** $V : \Delta \to \{0, 1\}$ as:

- $V(\bot) = 0$
- $V(\varphi) = v(\varphi) \quad \text{if } \varphi \in \mathrm{PV}$
- $V(\varphi \land \psi) = \min\{V(\varphi), V(\psi)\}$
- $V(\varphi \lor \psi) = \max\{V(\varphi), V(\psi)\}$
- $V(\varphi \to \psi) = \mathbb{1}_{V(\varphi) \leq V(\psi)}$

Semantics of Classical Propositional Calculus

Definition

We say a formula φ is classically valid and write it as $\vDash \varphi$ whenever for every valuation v we have $V(\varphi) = 1$.

Semantics

Heyting Algebras

Definition

A partial order $\{H, \leq\}$ is a **Heyting algebra** if:

- Every two elements $a, b \in$ H have a supremum ($a \cup b$) and an infimum ($a \cap b$) in H.
- Every two elements $a, b \in H$ have a relative pseudo complement ($a \longrightarrow b$), which is the greatest $c \in H$ such that $a \cap c \leq b$.
- H has both top (1) and bottom (0) elements.

Semantics

Semantics

Definition

Given a Heyting algebra $\mathscr{H} = \{H, \leq, \cup, \cap, 0, 1, \Rightarrow\}$, an **intuitionistic valuation** is a function from **PV** to H.

Definition

Given a Heyting algebra $\mathscr{H} = \{H, \leq, \cup, \cap, 0, 1, \Rightarrow\}$ and a valuation v, we define the **value function** $V : \Delta \to H$ as:

- $V(\bot) = 0$
- $V(\varphi) = v(\varphi) \quad \text{if } \varphi \in \mathrm{PV}$
- $V(\varphi \wedge \psi) = V(\varphi) \cap V(\psi)$
- $V(\varphi \vee \psi) = V(\varphi) \cup V(\psi)$
- $V(\varphi \to \psi) = V(\varphi) \Rightarrow V(\psi)$

Semantics

Definition

We write $\vDash \varphi$ whenever we have that $V(\varphi) = 1$, for every Heyting algebra $\mathscr{H}$ and every valuation v.

Semantics

Theorem

$\vdash \varphi$ *if and only if* $\models \varphi$.

Semantics

Non-redundancy of the Law of Excluded Middle

Theorem

$\nvDash p \vee \neg p$

Proof.

$\square$

Semantics

Theorem

$\nvdash \neg\neg p \rightarrow p$

Proof.

$\square$

Glivenko's Theorem

Theorem

A formula φ is classically valid if and only if $\neg\neg\varphi$ is intuitionistically valid.

That’s all

Questions?