Bézout's Theorem

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- Sheaves, schemes, and beyond!
- Focus today: plane curves.

Two Curves

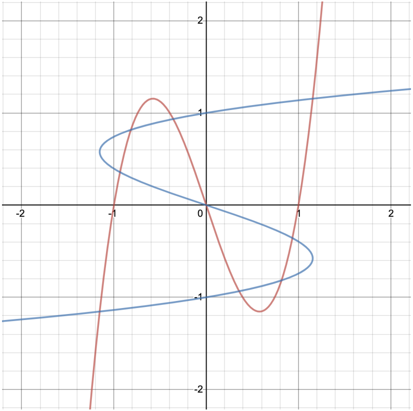

Figure: Two cubic curves.

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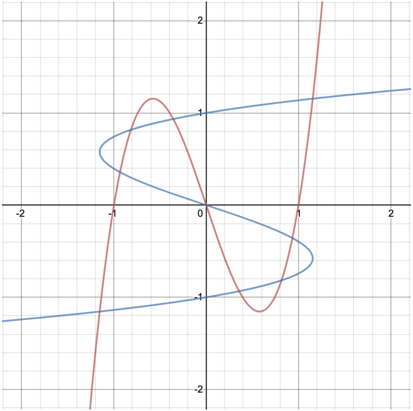

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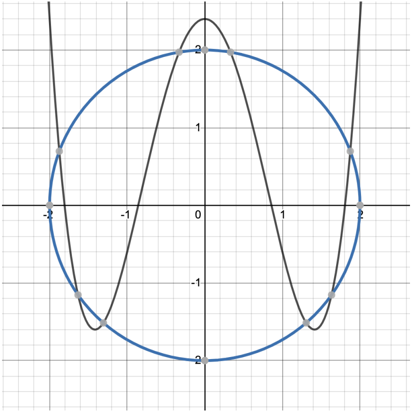
$$3 \times 3 = 9$$

Another Example



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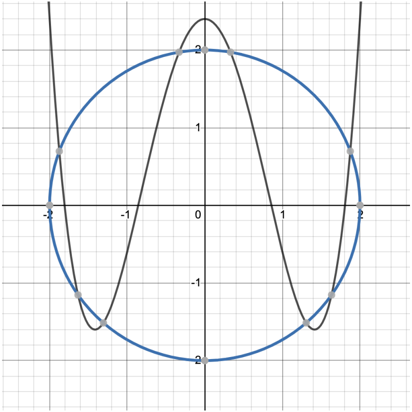

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$$2 \times 4 = 8$$

Is this a counterexample?

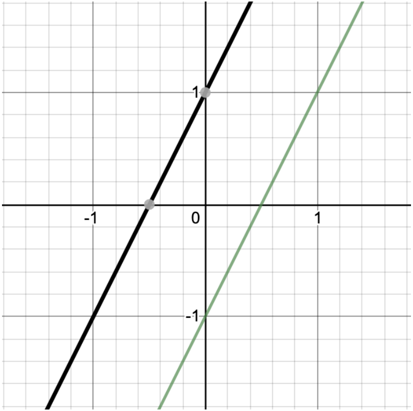

Figure: Parallel lines.

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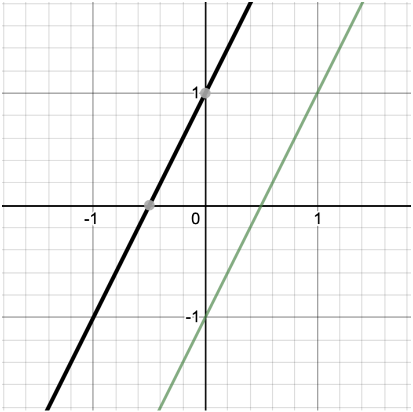

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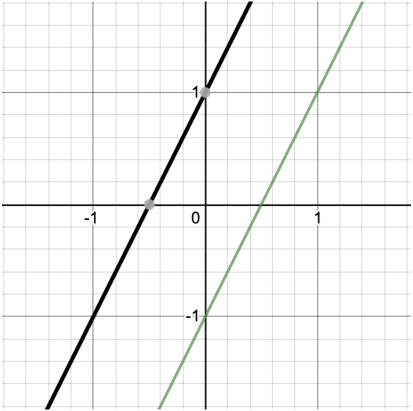

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- Look at **projective space**.

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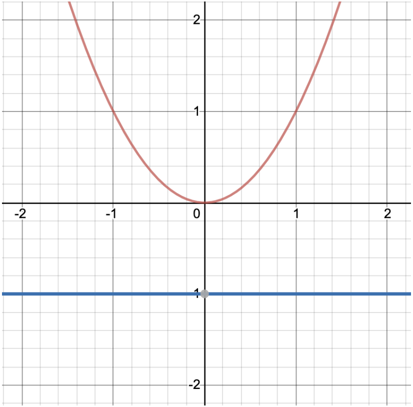

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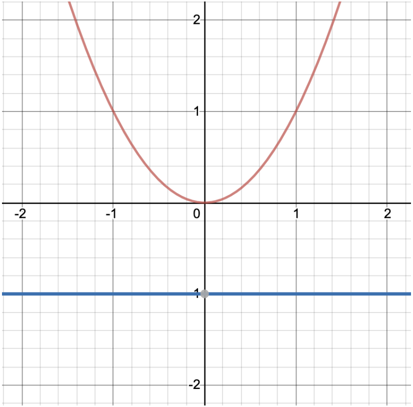

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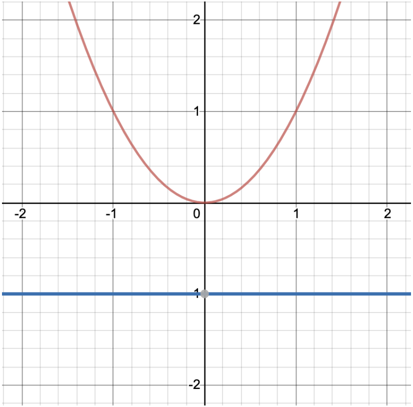

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- Where do these intersect?
- They don't "intersect at infinity."
- $x^2 = -1$
  $\implies x = \pm\sqrt{-1}$.
- Look at **complex** projective space.

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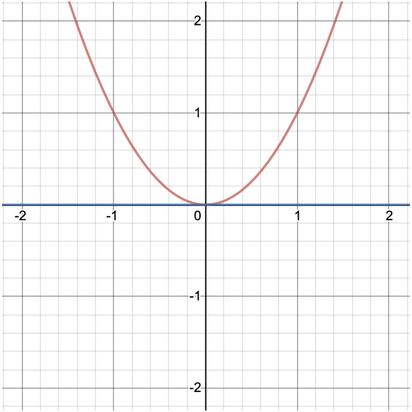

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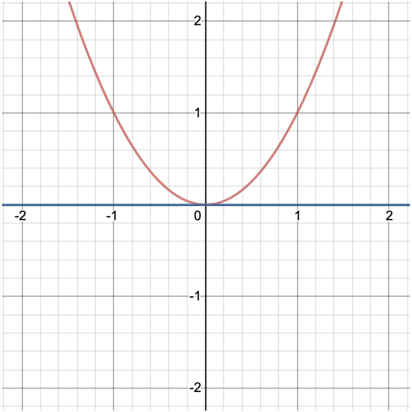

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- x^2 has a double root at 0.

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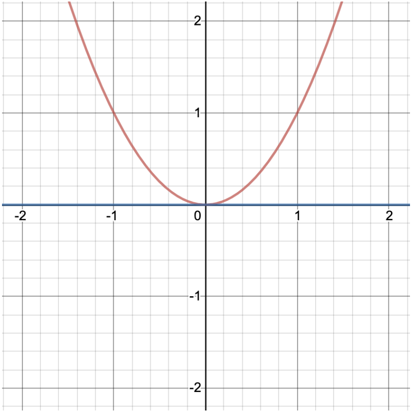

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- Count multiplicity.

Bézout's Theorem

Theorem (Bézout)

Let C_1 and C_2 be projective plane curves of degree d_1 and d_2, respectively.

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Let C_1 and C_2 be projective plane curves of degree d_1 and d_2, respectively. Suppose that C_1 and C_2 do not share a common component. Then C_1 and C_2 intersect in exactly $d_1 d_2$ points, counted with multiplicity.

The Projective Plane

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We define an equivalence relation $\sim$ on $\mathbb{C}^3 \setminus \{0\}$ by $(a_1, b_1, c_1) \sim (a_2, b_2, c_2)$ if and only if $(a_1, b_1, c_1) = (\lambda a_2, \lambda b_2, \lambda c_2)$ for some $\lambda \in \mathbb{C}^*$.

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Definition

The *projective plane* is

$$\mathbb{P}^2 := \mathbb{P}^2(\mathbb{C}) := (\mathbb{C}^3 \setminus \{0\})/\sim := \{[x : y : z] : (x, y, z) \neq 0\}.$$

Points on the projective plane

Note that the plane $\mathbb{A}^2_{\mathbb{C}} := \mathbb{C}^2$ sits inside $\mathbb{P}^2$ as

$$\{[a : b : 1] \in \mathbb{P}^2 : a, b \in \mathbb{C}\}.$$

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$$\{[a : b : 1] \in \mathbb{P}^2 : a, b \in \mathbb{C}\}.$$

Every other point looks like $[a : b : 0]$ (points at infinity).

Homogeneous Polynomials

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- $x + 3y - 2z$
- $x^2 + y^2 - z^2$
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If f is a homogeneous polynomial of degree d, then for any $\lambda \in \mathbb{C}$,

$$f(\lambda x, \lambda y, \lambda z) = \lambda^d f(x, y, z)$$

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Definition

A *projective plane curve* is a set of the form

$$V(f) := \{[a : b : c] \in \mathbb{P}^2 : f(a, b, c) = 0\},$$

where $f \in \mathbb{C}[x, y, z]$ is homogeneous and nonzero. The degree of f is called the *degree* of the curve.

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We have two parallel lines $y = 2x + 1$ and $y = 2x - 1$. Where do they intersect? First, homogenize to get $y - 2x - z$ and $y - 2x + z$. These intersect when

$$y - 2x - z = y - 2x + z = 0.$$

We get $2z = 0 \implies z = 0$. If $z = 0$, then we have $y = 2x$. If $x = 0$, then $(x, y, z) = (0, 0, 0)$, which doesn't give a point on the projective plane. So x is nonzero, and the intersection is at $[x : 2x : 0] = [1 : 2 : 0]$.

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- In general, if $f(x, y, z)/g(x, y, z) \in \mathbb{C}(x, y, z)$ is a rational function, f/g is not well-defined at a point $[a : b : c] \in \mathbb{P}^2$.

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- However, if f and g are both homogeneous of the same degree d, then for any $\lambda \in \mathbb{C}^*$,

$$\frac{f(\lambda x, \lambda y, \lambda z)}{g(\lambda x, \lambda y, \lambda z)} = \frac{\lambda^d f(x,y,z)}{\lambda^d g(x,y,z)} = \frac{f(x,y,z)}{g(x,y,z)}.$$

Function Field of $\mathbb{P}^2$

Definition

The *function field*, or the *field of rational functions*, of $\mathbb{P}^2$ is

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Examples

$$\frac{1}{1}, \quad \frac{x}{z}, \quad \frac{x^3 + y^3}{x^2y + y^2z + z^2x}.$$

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Examples

For $p = [0 : 0 : 1]$, $\frac{0}{1}, \frac{x}{z}, \frac{y}{z}, \frac{z}{z}, \frac{x^2}{x^2+z^2} \in \mathcal{O}_{\mathbb{P}^2,p}$.
$\frac{x}{y}$ is **NOT** in $\mathcal{O}_{\mathbb{P}^2,p}$ in this case.

Defining Intersection Multiplicity

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"Usually" this will be 0 or 1. We have

$$I_p(C_1, C_2) \geq 1 \iff p \in C_1 \cap C_2.$$

A Technical Detail

With $p = [a : b : 1] \in \mathbb{A}^2 \subset \mathbb{P}^2$ as above, we have an isomorphism

$$\begin{aligned} \varphi : \mathcal{O}_{\mathbb{P}^2,p} &\to \mathbb{C}[x,y]_{\mathfrak{m}_p} \\ h(x,y,z) &\mapsto h(x,y,1) \\ H(x/z, y/z) &\leftarrow\!\shortmid H(x,y) \end{aligned}$$

where

$$\mathfrak{m}_p := \{H \in \mathbb{C}[x,y] : H(p) = 0\} = (x-a, y-b) \subset \mathbb{C}[x,y]$$

is the maximal ideal corresponding to p. The forward map is *dehomogenization*, and the inverse map is called *homogenization*. This makes calculations slightly less cumbersome.

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We have $a + bx + cy + dx^2 + exy + \cdots \equiv a \mod (x, y)$. So only scalars, a 1-d vector space, so

$$I_p(C_1, C_2) = 1.$$

A Non-Intersection

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In general, we see that if either f or g does not vanish at p, then $(f, g) = \mathcal{O}_{\mathbb{P}^2, p}$, the quotient is trivial, and $I_p(C_1, C_2) = 0$.

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Examples

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so in $\mathbb{C}[x, y]_{\mathfrak{m}_p}/(f, g)$, we are left with elements of the form

$$\frac{au + b}{cu + d},$$

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so in $\mathbb{C}[x, y]_{\mathfrak{m}_p}/(f, g)$, we are left with elements of the form

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with $d \neq 0$. But

$$\frac{au + b}{cu + d} \cdot \frac{-cu + d}{-cu + d} = \frac{-acu^2 + (ad - bc)u + bd}{d^2 - c^2u^2} = \frac{ad - bc}{d^2}u + \frac{b}{d}.$$

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So every element in $\mathbb{C}[x, y]_{\mathfrak{m}_p}/(f, g)$ is of the form $au + b$ for some $a, b \in \mathbb{C}$.

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Examples

So every element in $\mathbb{C}[x,y]_{\mathfrak{m}_p}/(f,g)$ is of the form $au+b$ for some $a,b\in\mathbb{C}$. In fact, 1 and u are linearly independent over $\mathbb{C}$: suppose $au+b=0$. Then $ax+b\in(f,g)$, which is impossible (look at f and g again). So $I_p(C_1,C_2)=2$.

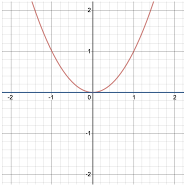

Figure: A parabola and a line.

Total Number of Intersections

The total number of intersections between C_1 and C_2 can then be counted as

$$\sum_{p \in \mathbb{P}^2} I_p(C_1, C_2).$$

This sum is finite as long as C_1 and C_2 don't share a common component.

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Bézout's Theorem

Let C_1 and C_2 be projective plane curves of degree d_1 and d_2, respectively. Suppose that C_1 and C_2 do not share a common component. Then

$$\sum_{p \in \mathbb{P}^2} I_p(C_1, C_2) = d_1 d_2.$$

Thank You

Sorry there's not time for a proof, but thank you for listening!
Questions?