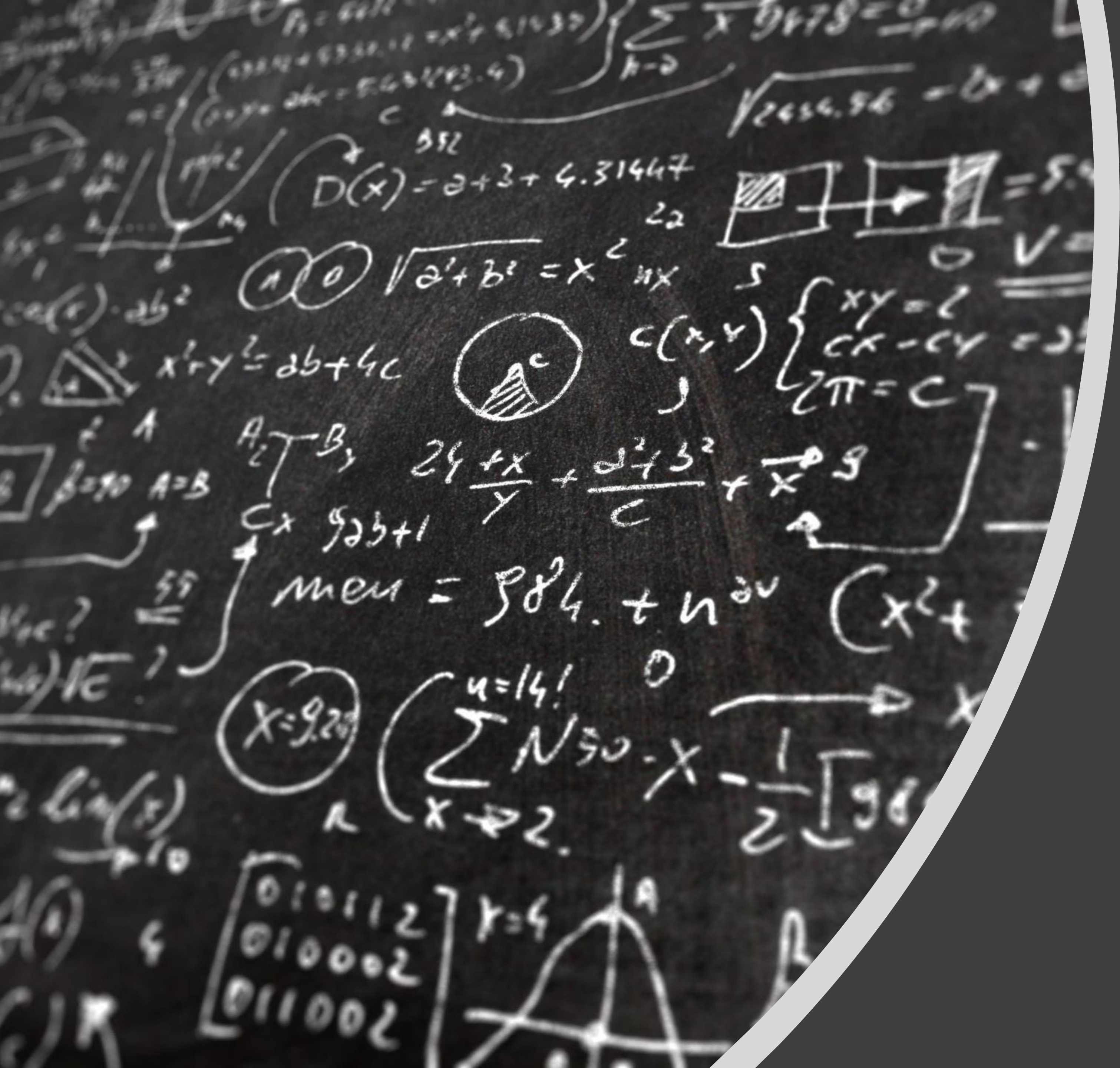


Nonlinear Dynamics & Chaos

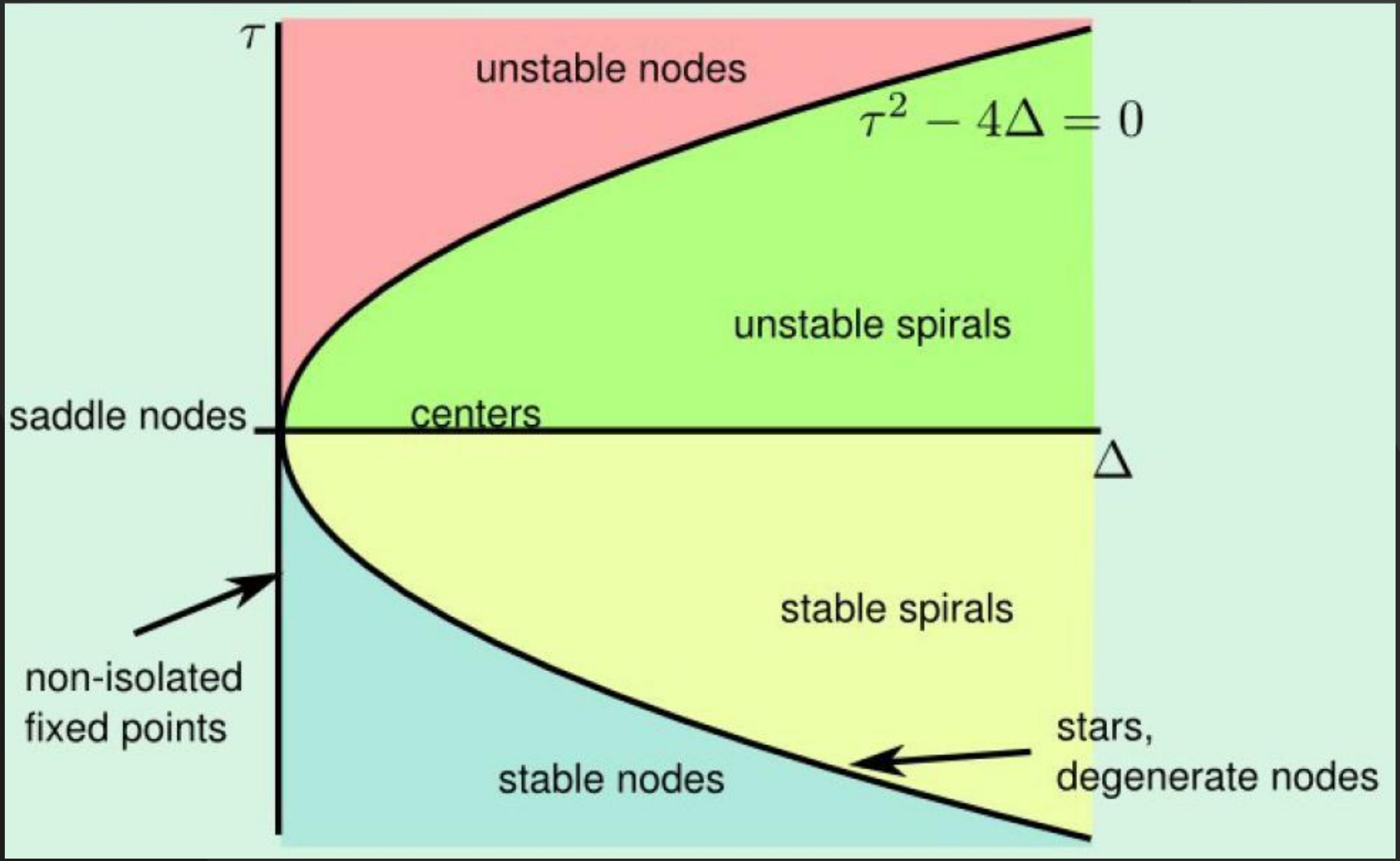
Andrew White

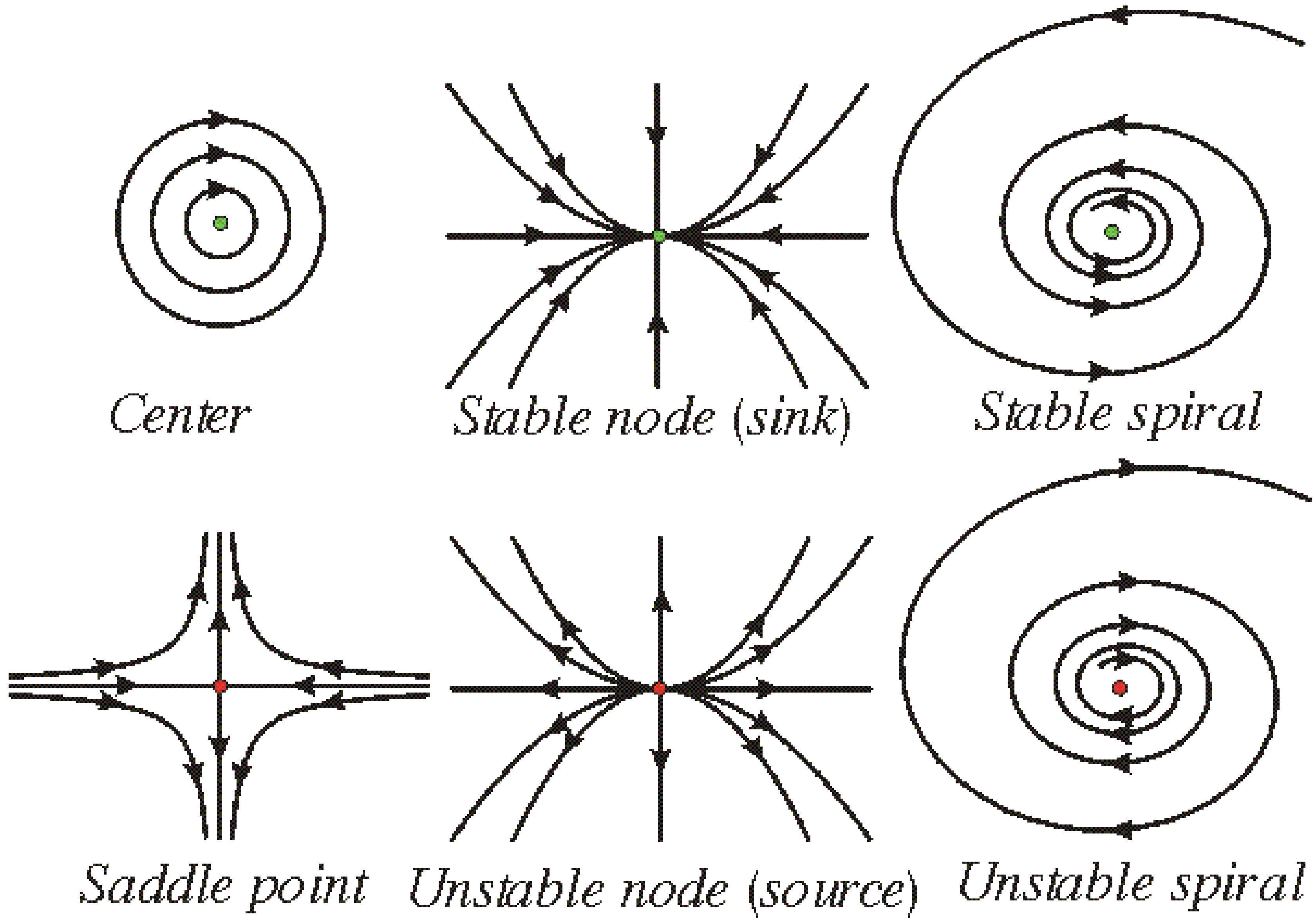
Linear Dynamics

- All systems of the form $\boldsymbol{x}' = A\boldsymbol{x}$
- Very powerful, linear systems of ODE's all orders can be represented in this form
- Solving for Equilibria:
- $\Leftrightarrow \boldsymbol{x}' = \mathbf{0} \Leftrightarrow A\boldsymbol{x} = \mathbf{0}$
- Always has $\mathbf{0}$ as a solution, if matrix is singular, could have more solutions

Linear Dynamics in 2D

- Fixed point can be classified by sign of eigenvalues of A.
- $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $A\boldsymbol{\zeta} = \lambda\boldsymbol{\zeta} \Leftrightarrow A\boldsymbol{\zeta} - \lambda\boldsymbol{\zeta} = \mathbf{0} \Leftrightarrow (\mathrm{A} - \lambda I)\zeta = 0$
- Det$(\mathrm{A} - \lambda I) = 0 = \begin{vmatrix} a - \lambda & b \\ c & d - \lambda \end{vmatrix} = \lambda^2 - (a + d)\lambda + (ad - bc)$
- Let trace$(\mathrm{A}) = \tau = a + d$, det$(\mathrm{A}) = \Delta = \mathrm{ad} - \mathrm{bc}.$
- Then, $\lambda_1 = \frac{\tau + \sqrt{\tau^2 - 4\Delta}}{2}$, $\lambda_2 = \frac{\tau - \sqrt{\tau^2 - 4\Delta}}{2}$


τ
unstable nodes
$\tau^2 - 4\Delta = 0$
unstable spirals
saddle nodes
centers
Δ
stable spirals
non-isolated
fixed points
stars,
degenerate nodes
stable nodes


Center
Stable node (sink)
Stable spiral
Saddle point
Unstable node (source)
Unstable spiral

Nonlinear Systems

- Generally have multiple equilibria but can use a similar framework to characterize the solutions
- For 2D, systems are of the form: ${x'}_1 = f_1(x_1, x_2); {x'}_2 = f_2(x_1, x_2)$.
- Then, the behavior of the system near an equilibrium (in most cases) can be determined by the signs of the eigenvalues of the Jacobian matrix at that point.
- $J(u^*) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(u^*) & \frac{\partial f_2}{\partial x_1}(u^*) \\ \frac{\partial f_2}{\partial x_1}(u^*) & \frac{\partial f_2}{\partial x_1}(u^*) \end{bmatrix}$

More Generally:

- Hartman–Grobman theorem: Consider a system evolving in time with state $u(t) \in \mathbb{R}^n$ that satisfies the differential equation $\frac{du}{dt}$ for some smooth map $f\colon \mathbb{R}^n \to \mathbb{R}^n$. Suppose that has a hyperbolic equilibrium state $u^* \in \mathbb{R}^n$, that is $f(u^*) = 0$ and the Jacobian matrix of f at u^* has no eigenvalue with real part equal to 0. Then there exists a neighborhood N of u^* and a homeomorphism $h\colon N \to \mathbb{R}^n$ s.t. $h(u^*) = 0$ and s.t. in the neighborhood N the flow of $\frac{du}{dt} = f(u)$ is topologically conjugate by the continuous map $U = h(u)$ to the flow of its linearization $\frac{dU}{dt} = JU$.

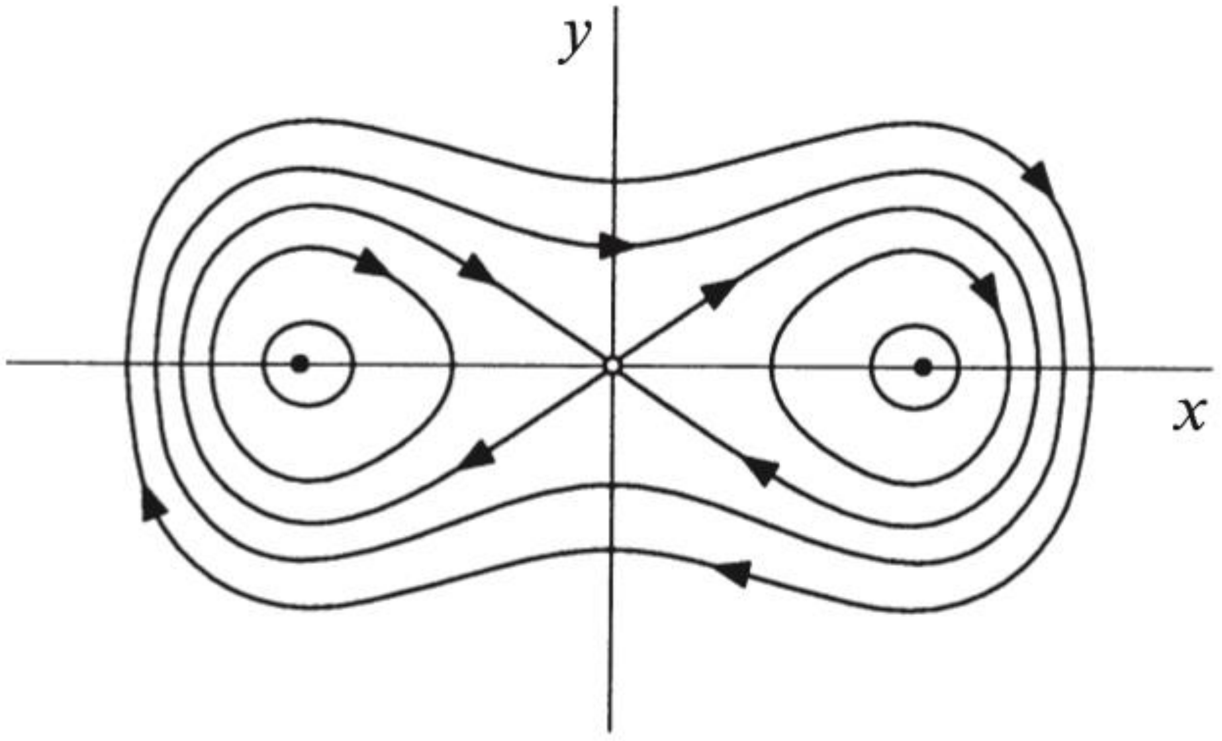
Indeterminate Cases

- Stars, degenerate nodes, centers, non-isolated fixed points are indeterminate
- However, if we just care about stability generally, then we can restrict our attention to only centers and non-isolated fixed points.
- Non-isolated fixed points don't occur very often (0 eigenvalue $\Rightarrow J$ degenerate)
- Hopefully, I've convinced you we should be worried about indeterminacy of centers

Energy Method

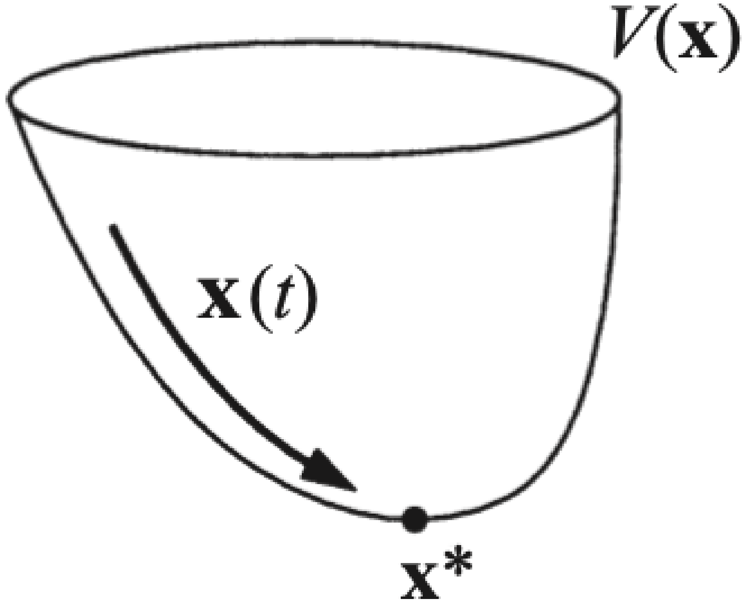
- If we can find a conserved quantity, which we call energy, this guarantees that centers predicted in the linearized system are centers in the nonlinear system.
- Consider the equation of motion for a particle moving in a double well potential $x'' = x - x^3$, which can be written equivalently as: $x' = y;\ y' = x - x^3$.
- The system has equilibria at $(x^*, y^*) = (0,0), (\pm 1,0)$. The Jacobian is $J(x, y) = \begin{pmatrix} 0 & 1 \\ 1 - 3x^2 & 0 \end{pmatrix}$.
- At $(\pm 1,0)$, $\tau = 0$ and $\Delta = 2$, so the equilibria are predicted to be centers.
- How do we know? $x'' = x - x^3 \Longrightarrow x'(x'') = x'x - x'x^3 \Longrightarrow x'x'' - x'x - x'x^3 = 0$
- By the chain rule: $\frac{d}{dt}\left[\frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4\right] = 0$, so $E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{4}x^4 = C$.

Phase Plane of the System



Liapunov Functions

- Another way to rule out ambiguity, if we can find one, then no centers
- Suppose $\boldsymbol{x}' = f(\boldsymbol{x})$ has an equilibrium $\boldsymbol{x}^*$. A Liapunov function $V(\boldsymbol{x})$ is continuously differentiable, real-valued, positive definite and satisfies $V'(\boldsymbol{x}) < 0$ for every $\boldsymbol{x}$ out of equilibrium.
- Proof: beyond the scope of this presentation
- Graphically:



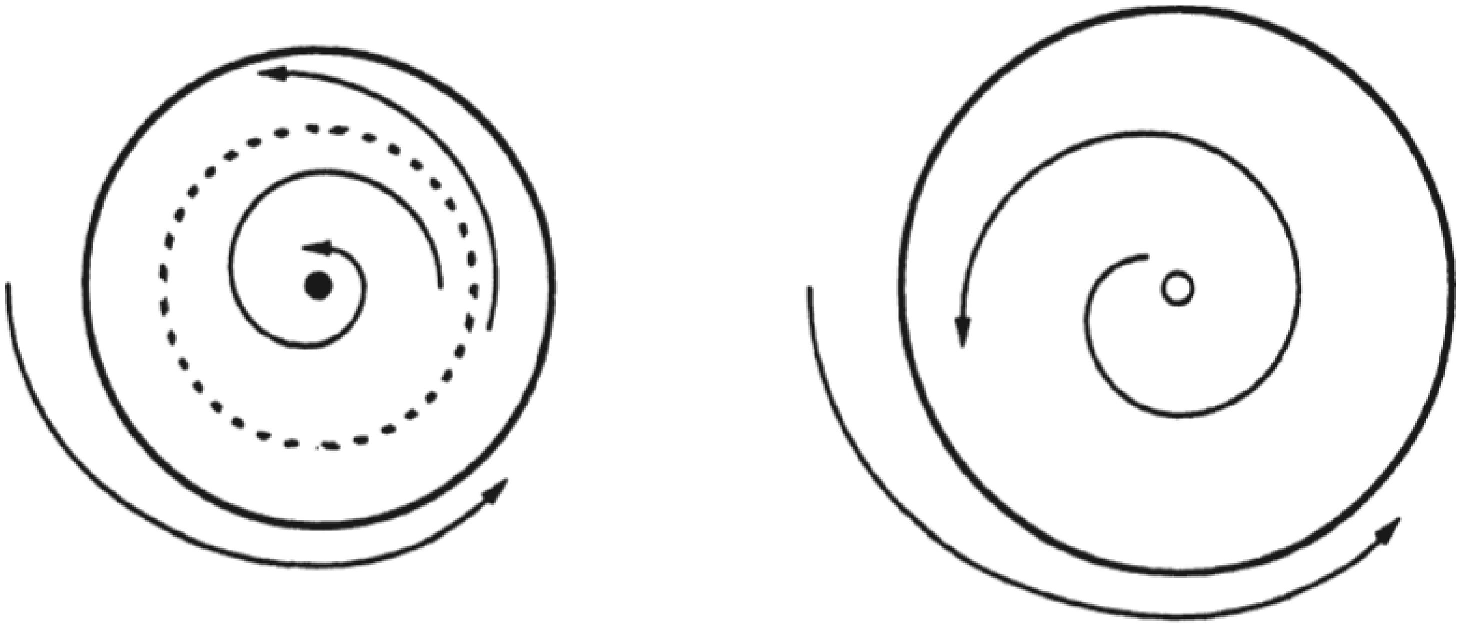
Bifurcations

- Occur when the topological structure of phase portrait changes as a parameter is varied
- Include changes in the number or stability of equilibria
- These types can normally be detected using the Jacobian matrix.
- Example: consider the system $x' = \mu x - y + xy^2; y' = x + \mu y + y^3$.
- By inspection, $(0,0)$ is an equilibrium. We have that

$$J(0,0) = \begin{pmatrix} \mu & -1 \\ 1 & \mu \end{pmatrix}$$

- $\tau = 2\mu$ and $\Delta = u^2 + 1 > 0$. From our picture, we see that as μ increases, the equilibrium changes from a stable spiral to an unstable spiral.

Phase Plane of the System:



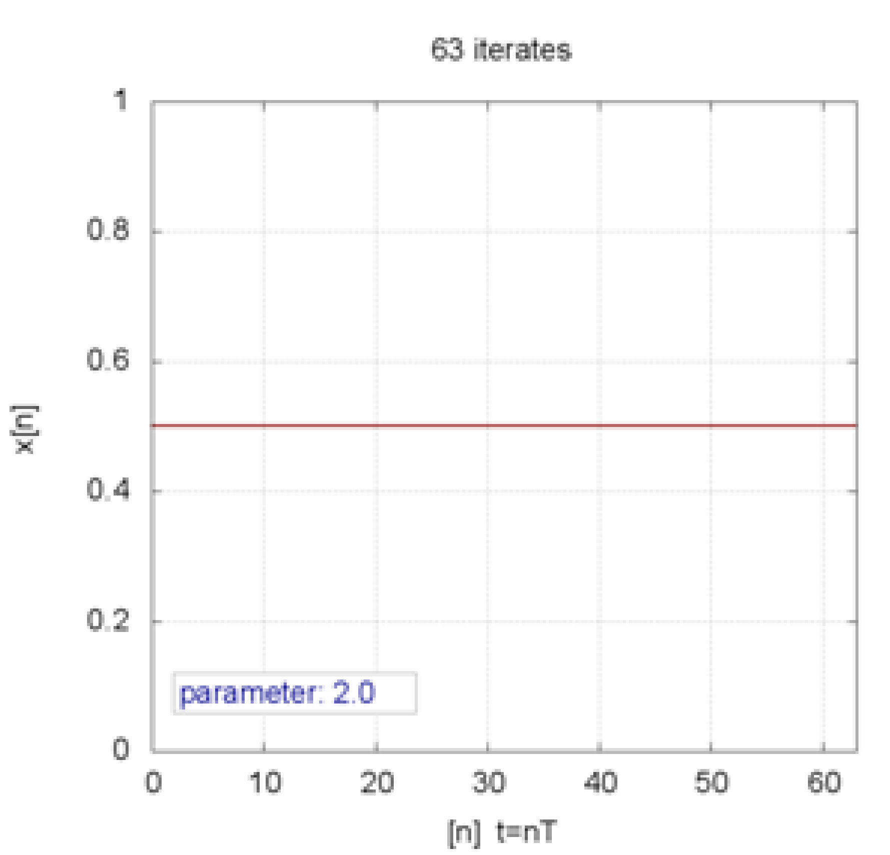
$\mu < 0$ $\mu > 0$

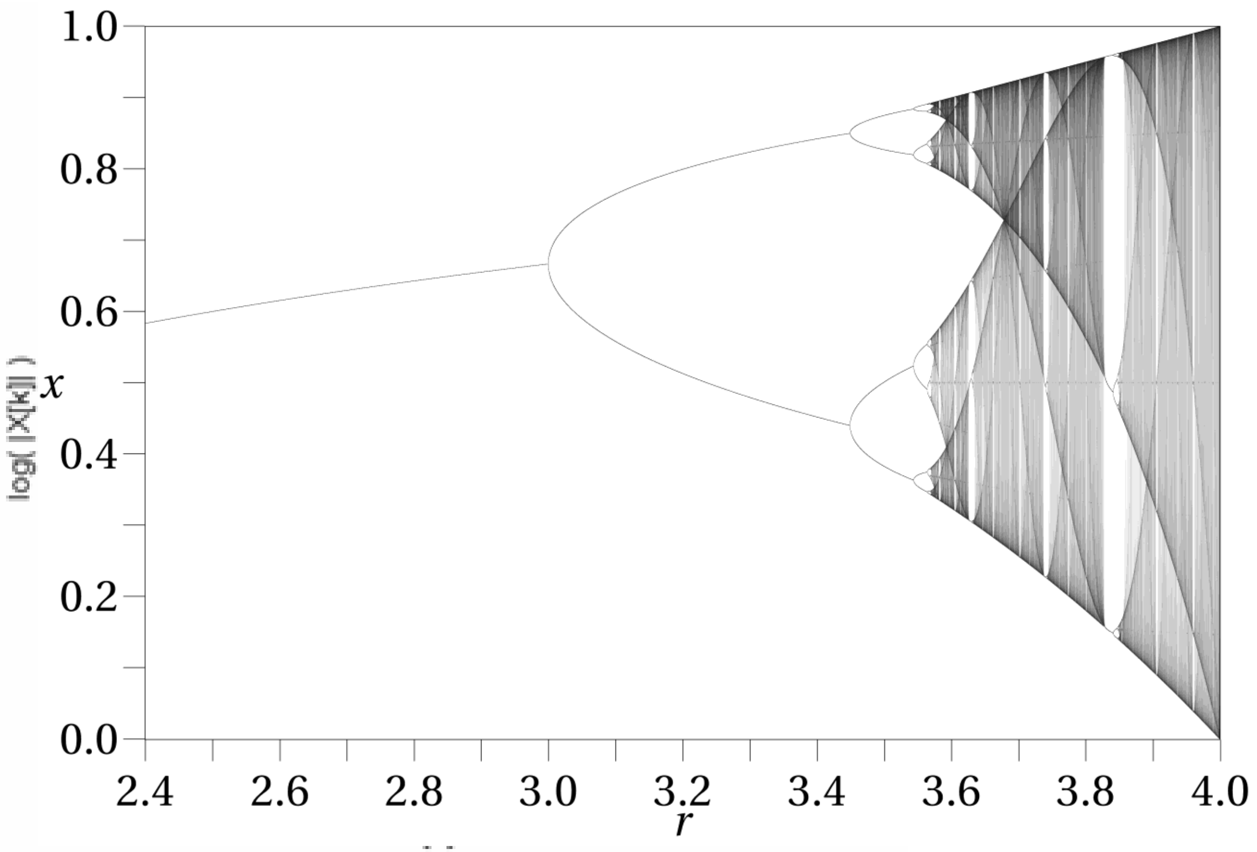
Chaos

- Definition: (1) Aperiodic long-term behavior in a (2) deterministic system that exhibits (3) sensitive dependence on initial conditions.
    1. There exists trajectories that do not settle to fixed points, periodic orbits, or quasiperiodic orbits (these only occur on the torus)
    2. No random or noisy input parameters
    3. Nearby trajectories separate exponentially fast.
- For continuous systems, can only occur in 3D and up.
- For discrete systems, can occur in 1D.

The Logistic Map

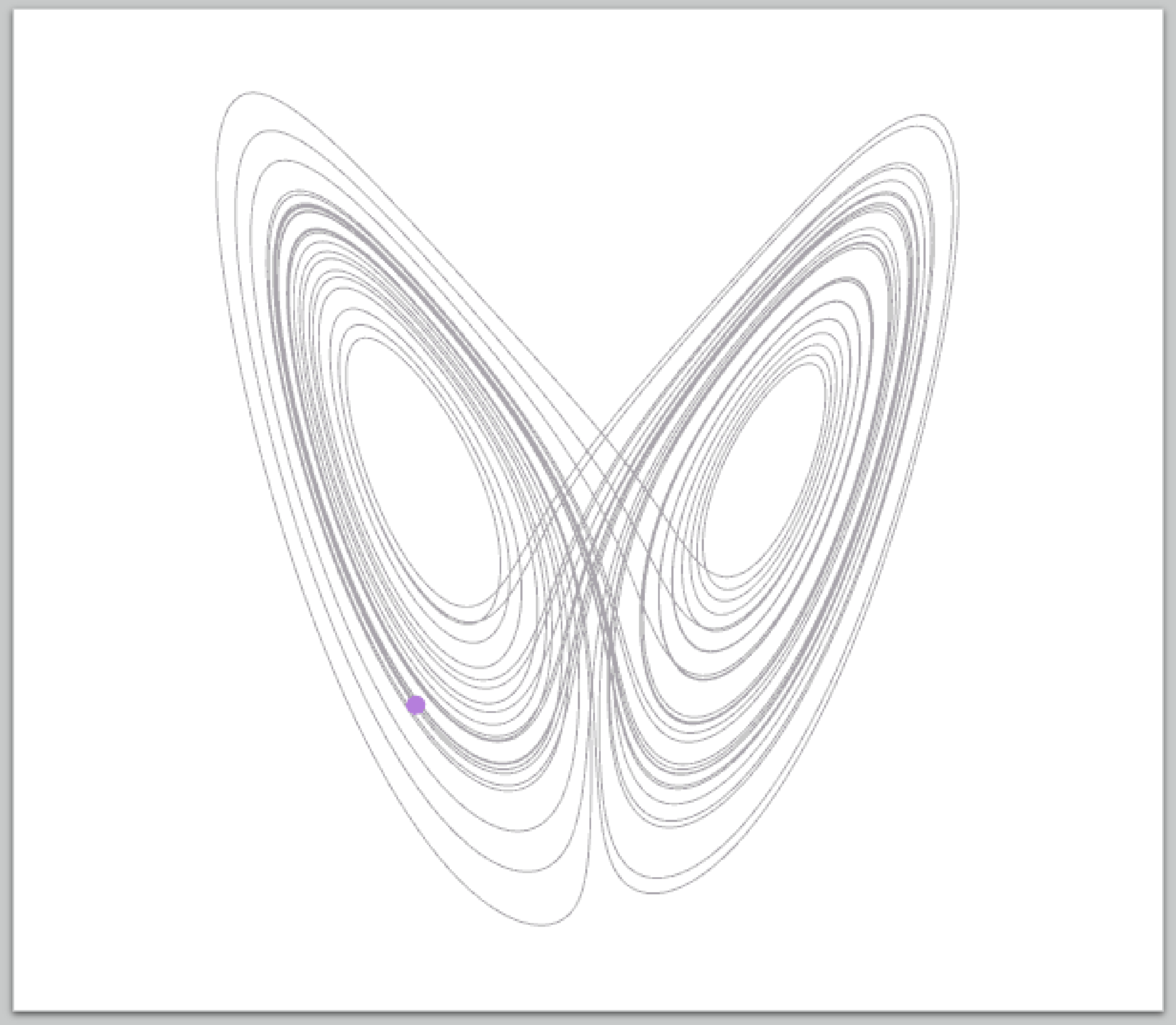
- $x_{n+1} = rx_n(1 - x_n)$





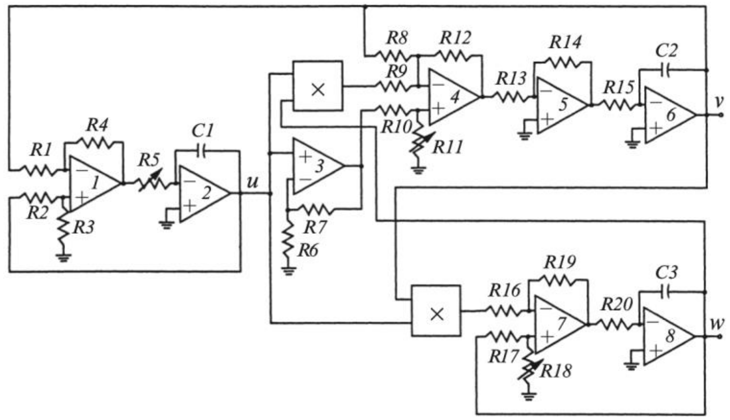
Lorenz System

- $x' = \sigma(y - x)$
- $y' = rx - y - xz$
- $z' = xy - bz$



Application: Synchronized Circuits

$$\dot{u} = \sigma(v - u)$$
$$\dot{v} = ru - v - 20uw$$
$$\dot{w} = 5uv - bw$$



$$\dot{u}_r = \sigma(v_r - u_r)$$
$$\dot{v}_r = ru(t) - v_r - 20u(t)w_r$$
$$\dot{w}_r = 5u(t)v_r - bw_r$$