MATH 101: ALGEBRA I
MIDTERM EXAM

Name ____

Problem	**Score**
1	
2	
3	
4	

Total _____________

Date: 11 October 2016.

Problem 1. Let G be a group. Indicate if the following statements are true or false. If true, give a proof; if false, give an explicit counterexample.

(a) If $H, H' \trianglelefteq G$ and $G/H \simeq G/H'$, then $H \simeq H'$.

(b) If $H, H' \trianglelefteq G$ and $H \simeq H'$, then $G/H \simeq G/H'$.

(c) If K, K' are groups and $G \times K \simeq G \times K'$, then $K \simeq K'$.

Problem 2. Let R be a Euclidean domain with norm N.

(a) Let

$$m = \min(\{N(a) : a \in R, a \neq 0\}).$$

Show that every nonzero $a \in R$ with $N(a) = m$ is a unit in R.

(b) Deduce that a nonzero element of norm zero in R is a unit; show by an example that the converse of this statement is false.

(c) Let F be a field and let $R = F[[x]]$. Show that R is Euclidean. What does part (a) tell you about $R^\times$? What are the irreducibles in R, up to associates?

Problem 3. Let F be a field and let $V = \mathrm{Mat}_{2\times 3}(F)$ be the F-vector space of 2×3-matrices.

(a) The group $\mathrm{GL}_2(F)$ acts on V by left multiplication. For $M, M' \in V$, the relation $M \sim M'$ if and only if $M' = AM$ for some $A \in \mathrm{GL}_2(F)$ defines an equivalence relation on V.

What are the equivalence classes (i.e., the *orbits* of the action)?

(b) Show that this action $\mathrm{GL}_2(F) \circlearrowleft V$ induces an injective group homomorphism

$$\phi : \mathrm{GL}_2(F) \hookrightarrow \mathrm{Aut}_F(V).$$

(c) Under the isomorphism $\mathrm{Aut}_F(V) \simeq \mathrm{GL}_6(F)$ given by the basis of matrix units, describe ϕ explicitly.

Problem 4. For the purposes of this exercise, we say that an isomorphism of F-vector spaces is *natural* if it does not depend on a choice of basis.

Let F be a field and let V, W be finite-dimensional vector spaces over F. Show that there is a (well-defined) natural isomorphism of F-vector spaces

$$\phi : V^* \otimes_F W \xrightarrow{\sim} \mathrm{Hom}_F(V, W).$$