MATH 101: ALGEBRA I
FINAL EXAM

Name ______________________________

Problem	Score
1	
2	
3	
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5	
6	

Total __________

Date: Tuesday, 22 November 2016.

Problem 1. Let F be a field. For a group G written multiplicatively, recall that the group algebra $F[G]$ is the F-vector space with basis G and multiplication induced by the group law in G, extended F-linearly.

Let $G = S_3$, let $\tau = (1\ 2)$, and let $\alpha = 1 + \tau \in F[G]$.

(a) The element α acts F-linearly by left multiplication on $F[G]$:

$$\begin{aligned} T : F[G] &\to F[G] \\ \beta &\mapsto \alpha\beta \end{aligned}$$

Compute the matrix of T with respect to a basis of elements of G.

(b) Compute the minimal polynomial and characteristic polynomial of T.

(c) Let $B = F[G]$, let $I = \{\alpha\beta : \beta \in B\}$ be the right ideal of B generated by α. Observe that I and B/I are F-vector spaces, and compute $\dim_F I$ and $\dim_F(B/I)$.

Problem 2. Let $n \in \mathbb{Z}_{\geq 1}$. Let $A \in \mathrm{GL}_n(\mathbb{C})$ have n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$. Let $V = \mathrm{M}_n(\mathbb{C})$. Find the eigenvalues of the $\mathbb{C}$-linear map

$$\begin{aligned} T : V &\to V \\ M &\mapsto AMA^{-1}. \end{aligned}$$

Problem 3. Let p be an odd prime, and let $G = \mathrm{GL}_2(\mathbb{F}_p)$.

(a) Prove that a p-Sylow subgroup of G is cyclic, and exhibit a p-Sylow subgroup of G.

(b) Give two different reasons why every p-Sylow subgroup of G is conjugate to the one given in (a), at least one of which implies that any two *generators* of any two p-Sylow subgroups are conjugate.

(c) Show that there are exactly $p+1$ distinct p-Sylow subgroups in G.

Problem 4. Let k be a field and let $R = k[x, y]$ be the polynomial ring over k in the variables x, y.

(a) Show that the ideal $(x) \subseteq R$ generated by x is a projective R-module.

(b) Show that the ideal (x, y) generated by both x, y is *not* a projective R-module. *[Hint: Show that the surjective R-module homomorphism* $\phi : R^2 \to (x, y)$ *defined by* $\phi(e_1) = x$ *and* $\phi(e_2) = y$ *does not split.]*

Problem 5. Let $R = \mathbb{Z}[i]$ where $i^2 = -1$.

(a) Compute a generator of the ideal $(3 + 11i, 1 + 3i) \subseteq R$.

(b) Let M be the R-module generated by x_1, x_2, x_3 subject to the relations

$$(i+1)x_2 + (i-1)x_3 = 0$$
$$6x_1 + (3i-1)x_2 - (i+9)x_3 = 0$$

Compute the rank of M and the invariant factors of the torsion submodule $\mathrm{Tor}(M)$.

Problem 6. Let R be a commutative ring and let M, N be R-modules.

(a) State the universal property of $M \otimes_R N$.

(b) Suppose that R is a domain with field of fractions F, and that $N \subseteq M$ is an R-submodule such that M/N is a torsion R-module. Show that the inclusion $N \hookrightarrow M$ induces an F-vector space isomorphism

$$N \otimes_R F \xrightarrow{\sim} M \otimes_R F.$$