MATH 101: GRADUATE LINEAR ALGEBRA
WEEKLY HOMEWORK #1

Problem W1.1. Let V be a finite-dimensional vector space over a field F, and let $\phi\colon V \to V$ be an F-linear endomorphism of V.

(a) Show that there exists $m \geq 0$ so that $\operatorname{img}(\phi^m) \cap \ker(\phi^m) = \{0\}$.

(b) Now suppose that $\phi^2 = 0$. Show that the rank of ϕ is at most $(\dim V)/2$, and that there is a(n ordered) basis β for V such that $[T]_\beta$ has the block form

$$\begin{pmatrix} O & A \\ O & O \end{pmatrix}$$

i.e., has zeros in all blocks except possibly the upper right-hand corner.

Problem W1.2. Let V, W be finite-dimensional vector spaces over a field F, and let $\phi\colon V \to W$ be an F-linear map.

(a) Show that there exists a basis β of V and a basis γ of W such that $[\phi]_\beta^\gamma$ is diagonal with diagonal entries in $\{0, 1\}$. What does the number of 1s along the diagonal tell you about ϕ?

(b) Suppose now that $W = V$. Recalling the result from daily homework, show that there exists a basis β of V such that the conclusion of (a) holds for $[\phi]_\beta$ if and only if $\phi^2 = \phi$.

Problem W1.3. Suppose $\operatorname{char} F \neq 2$. Let V be an F-vector space. Recall that $\phi \in \operatorname{End}_F(V)$ is a projection if $\phi^2 = \phi$.

Let $\phi, \psi : V \to V$ be projection maps.

(a) Show that $\phi+\psi$ is a projection if and only if $\phi\psi = \psi\phi = 0$ if and only if $\operatorname{img}\phi \subseteq \ker\psi$ and $\operatorname{img}\psi \subseteq \ker\phi$.

(b) If $\phi + \psi$ is a projection, show that $\operatorname{img}(\phi + \psi) = \operatorname{img}(\phi) \oplus \operatorname{img}(\psi)$ and $\ker(\phi + \psi) = \ker(\phi) \cap \ker(\psi)$.

Problem W1.4. Let V be an F-vector space with $n = \dim_F V < \infty$. Let $A, B \subseteq V$ be F-subspaces with $a = \dim A$ and $b = \dim B$ and suppose $V = A + B$. Let

$$S = \{f \in \operatorname{End}_F(V) : f(A) \subseteq A,\ f(B) \subseteq B\}.$$

Observe that $S \subseteq \operatorname{End}_F(V)$ is an F-subspace, and then express $\dim S$ in terms of n, a, b.

Problem W1.5. Let V, W be finite-dimensional F-vector spaces, let $X \subseteq W$ be an F-subspace, and let $\phi : V \to W$ be F-linear. Prove that

$$\dim \phi^{-1}(X) \geq \dim V - \dim W + \dim X.$$

Date: Assigned Friday, 15 September 2017; due Friday, 22 September 2017.