MATH 101: GRADUATE LINEAR ALGEBRA
WEEKLY HOMEWORK #5

Problem W5.1. Let R be a ring and let

$$(*) \qquad 0 \to M \xrightarrow{\psi} N \xrightarrow{\phi} Q \to 0$$

be a short exact sequence of (left) R-modules. A *retraction* of $(*)$ is an R-module homomorphism $\rho\colon N \to M$ such that $\rho \circ \psi = \mathrm{id}_M$.

Show that $(*)$ is split if and only if $(*)$ has a retraction.

Problem W5.2. Let k be a field, let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix} \subseteq \mathrm{M}_2(k)$ be the subring of upper-triangular matrices. Let $P = \begin{pmatrix} k & 0 \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & k \\ 0 & k \end{pmatrix}$. Show that P and Q are projective left R-modules that are not free.

Problem W5.3. Let R be a ring and let P be a projective (left) R-module that has R as a direct summand. Show that if $P \oplus R^m \simeq R^n$ with $n > m$, then P^{m+1} is free.

Problem W5.4. Let $f\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be a homomorphism of groups.

(a) Show that if f is surjective, then f is injective.
(b) Show that if f is injective, then $\operatorname{coker} f$ is a finite group.

Date: Assigned Wednesday, 25 October 2017; due Wednesday, 1 November 2017.