Analysis Certification Questions

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Introduction

Below, we have compiled a list of typical exam questions. These are not meant to be exhaustive, but you should be familiar with all of them prior to scheduling and taking your exam. We will certainly ask many of these questions during a typical exam and, since you have been supplied with these questions in advance — and can ask either of us about the answers prior to the exam — it will reflect badly on you if you can not provide succinct (and correct) answers. *In other words, we suggest you work out the solutions and practice presenting them at the blackboard in front of a critical audience of your peers prior to the exam.* In particular, if you are unable to give *polished* answers to questions labeled as "Basic", we reserve the right to terminate the exam and ask you to return in six weeks.

1 Real Analysis

1.1 Basic

1. Let $f : A \subset \mathbf{R} \to \mathbf{R}$ be a function. Give three criteria (ϵ-δ, open sets, sequences) for f to be continuous on A. Show that they are equivalent.

2. Define what it means for $f : A \subset \mathbf{R} \to \mathbf{R}$ to be uniformly continuous on A. Show by example, that f can be continuous without being uniformly continuous on A.

3. Suppose that $f : (0,1] \to \mathbf{R}$ is continuous. Explain why f is uniformly continuous if and only if f can be extended to a continuous function on $[0,1]$.

4. If $f : [a,b] \to \mathbf{R}$ is continuous, then what sort of set is $f\big([a,b]\big)$. What does this have to do with the "Extreme Value Theorem" we teach our students in first semester calculus?

5. Let $\{\, f_n \,\}$ be a sequence of functions from $A \subset \mathbf{R}$ to $\mathbf{R}$. Define what it means for $\{\, f_n \,\}$ to converge pointwise to a function f on A. Define and compare this to uniform convergence. For example,

   (a) Show by example that a pointwise convergent sequence need not converge uniformly.

   (b) Is the pointwise limit of continuous functions necessarily continuous? What if the convergence is uniform?

   (c) Prove that the uniform limit of continuous functions is continuous.

   (d) Prove that if each f_n is continuous on $[a,b]$ and if $f_n \to f$ *uniformly* on $[0,1]$, then
   $$\int_a^b f_n(x) \to \int_a^b f(x).$$
   What if the convergence is only pointwise (but f is still continuous)?

   (e) If f is the uniform limit of differentiable functions $\{\, f_n \,\}$ on $[a,b]$, then must f be differentiable? If not, what *additional* hypothesis is necessary?

6. Use your answer to question 5e to deal with the following. Suppose that
$$f(x) = \sum_{n=0}^{\infty} a_n x^n$$
converges for $|x| < 1$. Show that f is differentiable and that for all $|x| < 1$ we have
$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}.$$

7. Let $f(x) = \cos(x^2)$. What is $f^{(2006)}(0)$ equal to? (How about $f^{(2008)}(0)$?)

1.2 Average

1. Can a Riemann integrable function on $[a, b]$ have discontinuities? Can it have infinitely many?

2. Can you characterize when a bounded function on $[a, b]$ is Riemann integrable.

3. Is every closed and bounded subset of a metric space compact? (Give a proof or an explicit counterexample.)

4. Give three equivalent properties of compactness in a metric space.

5. Give an example of a closed and bounded subset of a complete metric space which is not compact.

6. Let X be a compact Hausdorff space and let $C(X)$ be the normed vector space of continuous functions on X with respect to the supremum norm. Show that $C(X)$ is a Banach space.

1.3 Harder

1. Let X be a compact Hausdorff space. Suppose that $K \subset C(X)$. Under what conditions is K compact (in the norm topology induced by the supremum norm on $C(X)$). (You may want to think about the Arzelà-Ascoli Theorem.)

2. Describe the Cantor set and some of its properties.

3. Let f be an increasing function on $[0, 1]$. Show that f is Riemann integrable.

4. Give an example of a function on $\mathbf{R}$ which is continuous at each irrational and discontinuous at each rational. Can you find an example of a function which is continuous at each rational and discontinuous at each irrational? Why not? (See [Mun00, Chap. 8, Ex. 7].)

2 Complex Analysis

2.1 Basic

1. Define what is meant when we say that a function $f : D \to \mathbf{C}$ is holomorphic on an open connected subset $D \subset \mathbf{C}$ ([Rud87, Definition 10.2]).[1] (Hereafter, open connected subsets of the plane will be called *domains*.)

2. Where, if anywhere, are $f(z) = z^2$ and $f(z) = \frac{1}{1+z^2}$ holomorphic? What is the derivative in each of these cases? How about $h(z) = \bar{z}$, where $\overline{x + iy} := x - iy$ (for $x, y \in \mathbf{R}$)?

3. What are the Cauchy-Riemann equations? If the Cauchy-Riemann equations hold for f at a point $z_0 = x_0 + iy_0$, must f be complex differentiable at z_0? If not, what extra hypotheses must we add?

4. Give a careful statement of Cauchy's Theorem.

5. Give a careful statement of Cauchy's Integral Formula.

6. Explain why
$$f(z) := \frac{1}{1 + e^z}$$
has a convergent power series representation about $z = 0$. What is the radius of convergence?

7. What is the value of
$$\int_{|z|=1} \frac{\cos(z)}{z^4}\, dz?$$

8. Suppose that f is an entire function such that
$$f(\frac{1}{n}) = \frac{1}{n} \quad \text{for all } n \in \mathbf{Z}^+.$$
What can you say about f?

[1]We use the work *holomorphic* is the sense that Rudin does — of course many other (lessor?) authors prefer *analytic*. We prefer to reserve the word analytic for functions which are locally representable by power series as in [Rud87, Theorem 10.16] or [Con78, Theorem 2.8].

9. Why are the zeros of non-constant analytic functions isolated?

10. Suppose that f is a non-constant entire function. What can you *prove* about the range of f?

11. What are the different types of *isolated* singularities of analytic functions? Give examples of each type.

12. Suppose that f and g are entire functions such that $|f(z)| \leq |g(z)|$ for all z. By considering $h := f/g$, show that f must be a multiple of g.

2.2 Average

1. Does every analytic function on a domain D have an antiderivative? Why not? What if D is simply connected?

2. Use question 1 to show that every nonvanishing function f on a simply connected domain D has a logarithm. (That is, show there is a $g \in H(D)$ such that $f = e^g$.

3. What contour and meromorphic function would you use to evaluate

$$\int_0^\infty \frac{1}{1+x^3}\, dz?$$

   Do it.

4. Let $\{ f_n \}$ be a sequence of holomorphic functions on a domain D. Suppose that $f_n \to f$ uniformly on D. Must f be holomorphic? Why?

5. Suppose that f is an entire function such that $0 \leq |f(z)| \leq |z|^{\frac{1}{2}}$ for all z. What can you say about f?

2.3 Harder

1. What function and contour would you use to evaluate

$$\int_0^\infty \frac{\sin x}{x}\, dx?$$

   Do it.

2. Suppose that f is an entire function with a pole at ∞. Must f be a polynomial?

3. Question 2 can be generalized. Suppose that f analytic on $\mathbf{C}$ except for finitely many poles, and that f either has a removable singularity or a pole at ∞. Show that f is a rational function.

3 Measure Theory

3.1 Basic

1. What is a σ-algebra of sets in X?

2. What is a measure on X? Give some examples including Lebesgue measure on the real line.

3. Give a brief overview of how one defines Lebesgue measure.

4. Is $[a, b]$ Lebesgue measurable for all a and b? What is a Borel set? Are all Borel sets Lebesgue measurable?

5. What are Littlewood's three principles of real analysis?

6. Given a measure — say Lebesgue measure on $\mathbf{R}$ — give a brief overview of how we define the associated integral.

7. State the three fundamental convergence theorems: that is, state the Monotone Convergence Theorem, Fatou's Lemma and the Dominated Convergence Theorem. Why do we restrict to non-negative functions in the first two theorems? Show by example that the inequality in Fatou's Lemma can be a strict inequality.

8. What do you mean by $L^1(\mu)$ and $\|\cdot\|_1$? What do we have to do to make $\|\cdot\|_1$ a norm. Is $L^1(\mu)$ a Banach space (what is a Banach space)?

9. Does the Dominated Convergence Theorem imply convergence in L^1? If $f_n \to f$ in L^1, must it be the case that $f_n \to f$ pointwise almost everywhere? (Proof of counterexample please!) If each f_n is non-negative (almost everywhere) and if $f_n \to f$ in L^1, must f be non-negative almost everywhere?

10. What are the L^p spaces for $1 \leq p \leq \infty$. Are they Banach spaces?

11. What does Hölder's inequality say?

12. Let $(X, \mathcal{M}, \mu)$ be a measure space and $1 < p < q < \infty$? Is $L^p(\mu) \subset L^q(\mu)$? What about the other containment? What if $\mu(X) < \infty$? What if $X = \mathbf{N}$ and μ is counting measure (so that $L^p(\mu) = \ell^p$)? If $p < r < q$ and if $f \in L^p(\mu) \cap L^q(\mu)$, then under what conditions is $f \in L^r(\mu)$. See question 1 in Section 3.2 for more on this.

13. Are Riemann integrable functions Lebesgue integrable? In complex analysis courses, we often show that

$$\int_0^\infty \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

Does this imply that $f(x) = \sin(x)/x$ is Lebesgue integrable?

14. What does Fubini's Theorem say?

3.2 Harder

1. Let f be a measurable function on $\mathbf{R}$ and let

$$E = \{\, p \in R : 0 \leq p < \infty \text{ and } \|f\|_p^p < \infty. \,\}$$

What sort of set is E. For example is E necessarily connected, open or even closed? Can E be any connected subset of $(0, \infty)$? (See [Rud87, Chap 3, Ex 4] for even more interesting assertions.)

2. Are there Lebesgue measurable sets that are not Borel? Why?

3. Suppose that $f : [0, 1] \to [0, 1]$ is continuous. Let L be the set of y for which $\{\, x : f(x) = y \,\}$ has positive (Lebesgue) measure. Show that L has measure zero.

4 Functional Analysis

4.1 Basic

1. What is a bounded linear functional on a normed vector space X?

2. If X is any normed vector space, how do we know that there are enough bounded linear functionals on X to separate points?

3. How is the norm $\|\phi\|$ of a bounded linear functional ϕ on a normed vector space X defined?

4. Let $(X, \mathcal{M}, \mu)$ be a measure space and suppose that $\frac{1}{p} + \frac{1}{q} = 1$ for some $1 < p, q < \infty$. If $g \in L^q(\mu)$, then show that

$$f \mapsto \int_X f(x)g(x)\, d\mu(x)$$

is a bounded linear functional on $L^p(\mu)$. Are there any other bounded linear functionals on $L^p(\mu)$? What happens when $p = 1$ and $q = \infty$? What about when $p = \infty$?

5. What is a Hilbert space?

6. Suppose that H is a Hilbert space (over $\mathbf{C}$), and that $\phi : H \to \mathbf{C}$ is a continuous linear functional. What can you say about ϕ?

7. What does it mean to say that a linear operator T on a Hilbert space H is bounded? Give two definitions of the operator norm $\|T\|$: one involving a supremum and one involving an infimum. Show that they are indeed equal.

8. What is the adjoint, T^*, of a bounded linear operator T on H? Is T^* bounded? What can you say about the norm of T^*?

9. Show that every bounded operator on a Hilbert space has an adjoint.

10. What does the Closed Graph Theorem say?

4.2 Average

1. Suppose T is a bounded linear operator on a Hilbert space H. Show that $\|T^*T\| = \|T\|^2$.

2. Suppose that $T : H \to H$ is a bounded operator that is both injective and surjective. Must its inverse be bounded? Why — what theorem are you using?

3. Suppose that $\{T_n\}$ is a sequence of bounded linear operators on a Hilbert space H. Suppose that for each $h \in H$, $T_n h$ converges, say to Th. This defines a linear map T from H to itself. Must T be bounded? What theorem are you using?

4. Describe the dual of $C(X)$ (where X is a compact Hausdorff space).

5. Give an example of a weakly convergent sequence in a Banach space that is not norm convergent. Must such a sequence be norm bounded?

6. Define what it means for a linear operator T on a Hilbert space H to be *compact*. Is every compact operator bounded? Why or why not?

4.3 Harder

1. Show that the space X^* of bounded linear functionals ϕ on a normed vector space X is complete in the norm $\|\phi\|$ from above, even if X is not complete.

2. Show that every compact self-adjoint operator T on a Hilbert space H has an eigenvalue and state the corresponding Spectral Theorem for Compact Self-Adjoint Operators.

References

[Con78] John B. Conway, *Functions of one complex variable*, second ed., Graduate Texts in Mathematics, vol. 11, Springer-Verlag, New York, 1978. MR MR503901 (80c:30003)

[Mun00] James R. Munkres, *Topology: Second edition*, Prentice Hall, New Jersey, 2000.

[Rud87] Walter Rudin, *Real and complex analysis*, McGraw-Hill, New York, 1987.