Midterm for Math 103
Due Friday, November 14, 2008

Work on one side of $8\frac{1}{2} \times 11$ inch paper only. Start each problem on a separate page. (This last requirement can be waived for those LaTeX users whose very short and elegant solutions would result in an uncomfortable waste of paper.)

1. Let X be an uncountable set and let $\mathscr{M}$ be the connection of sets E in X such that either E or E^c is at most countable.

(a) Show that $\mathscr{M}$ is a σ-algebra.

(b) Show that

$$\mu(E) := \begin{cases} 1 & \text{if } E \text{ is uncountable, and} \\ 0 & \text{otherwise} \end{cases}$$

is a measure on $(X, \mathscr{M})$

(c) Describe the $\mathscr{M}$-measurable functions $f : X \to \mathbf{R}$ and their integrals.

ANS: The key to this problem is the a pairwise disjoint family $\{ E_i \}_{i=1}^{\infty}$ of sets in $\mathscr{M}$ can have at most one uncountable member: if E_1 is uncountable, then $E_i \subset E_1^c$ for all $i \geq 2$ and the latter is countable. After this, parts (a) and (b) are straightforward. The challenge is write things up neatly and elegantly.

Part (c) is a little trickier. You want to prove that if f is measurable, then there is exactly on $c \in \mathbf{R}$ such that $f^{-1}(c)$ is uncountable. It follows from the observation above, that there can be at most one such point. Let $\{ U_n \}$ be a countable basis of open sets for $\mathbf{R}$. Since exactly one of $f^{-1}(U_n)$ or $f^{-1}(U_n^c)$ is uncountable, let

$$B_n = \begin{cases} U_n & \text{if } f^{-1}(U_n) \text{ is uncountable, and} \\ U_n^c & \text{if } f^{-1}(U_n^c) \text{ is uncountable.} \end{cases}$$

Let $A = \bigcap_n B_n$. Notice that A contains at most one point: if x and y are distinct points in A, then there is a U_n which contains x and not y. Thus B_n can't contain both x and y. Thus if $f^{-1}(A)$ is uncountable, then we've proved the claim. Now observe that if $f^{-1}(C)$ and $f^{-1}(D)$ are both uncountable, then $f^{-1}(C \cap D)$ must be uncountable: to see this notice that $A \cup B$ is the disjoint union of $A \setminus B$, $A \cap B$ and $B \setminus A$. Thus

$$A = \bigcap_n B_n = \bigcap_n F_n \quad \text{where } F_n = B_1 \cap \cdots \cap B_n.$$

Now $f^{-1}(F_{n+1}) \subset f^{-1}(F_n)$ and $\mu(X) = 1 < \infty$. Thus

$$\mu(f^{-1}(A)) = \lim_n \mu(f^{-1}(F_n)) = 1.$$

Thus, $A \neq \emptyset$ and $f^{-1}(A)$ is uncountable. Hence $A = \{ c \}$ and $f(x) = c$ for μ-almost all x. But then

$$\int_X f(x)\, d\mu(x) = c \cdot \mu(X) = c.$$

2. Prove the "missing" results:

(a) Lemma 69: If $\{ f_n \}_{n=1}^{\infty}$ is a sequence of measurable functions which converges to a measurable function f in measure, then every subsequence also converges to f in measure.

(b) Theorem 70: Suppose that $\{ f_n \}_{n=1}^{\infty}$ is a sequence of measurable functions which converges to a measurable function f in measure and that $g \in \mathcal{L}^1(X)$ is such that, for each n, $|f_n(x)| \le g(x)$ for almost all x. Then prove that $f_n \to f$ in $L^1(X)$.

(Part (a) is really very straightforward. It is assigned as more of a hint for the second part than for any other reason.)

ANS: For part (a), fix $\epsilon > 0$ and let $a_n := E_n(\epsilon) = \{ x : |f_n(x) - f(x)| \ge \epsilon \}$. Then saying $f_n \to f$ in measure just means $a_n \to 0$. But if $\{ f_{n_k} \}$ is a subsequence, then $\{ a_{n_k} \}$ is too. But general nonsense says $a_{n_k} \to 0$. Thus $f_{n_k} \to f$ in measure. (This elegant argument is due to Aria Anavi.)

For part (b), suppose $f_n \not\to f$ in L^1. Then there is an $\epsilon_0 > 0$ and a subsequence, $\{ f_{n_k} \}$ such that $\|f_{n_k} - f\|_1 \ge \epsilon_0$ for all k. But $f_{n_k} \to f$ in measure (by part (a)). Thus, it has a subsequence $f_{n_{k_j}}$ converging pointwise almost everywhere to f. But then $f_{n_{k_j}} \to f$ in L^1 by the Dominated Convergence Theorem. But this contradicts the fact that $\|f_{n_{k_j}} - f\|_1 \ge \epsilon_0$ for all j.

3. If $f_n \to f$ pointwise almost everywhere, then must $f_n \to f$ in measure? Does you conclusion change if "almost everywhere" convergence is replace by pointwise convergence everywhere? What if $\mu(X) < \infty$? (Assume that each of f_n and f are measurable.)

ANS: This does not hold unless $\mu(X) < \infty$. Consider $f_n = \mathbb{1}_{[n,n+1]}$. Then $f_n \to 0$ pointwise (everywhere) on $\mathbf{R}$. However, $E_n(\epsilon) = 1$ for all n and all $0 < \epsilon < 1$. Thus, $f_n \not\to 0$ in measure. If $\mu(X) < \infty$, then $f_n \to f$ in measure by Egoroff's Theorem.

4. Counterexamples.

(a) Show that both the Monotone Convergence Theorem and Fatou's Lemma are false without the assumption that the f_n are nonnegative (at least almost everywhere).

(b) Show that Egoroff's Theorem fails if we drop that assumption that $\mu(X) < \infty$.

ANS: For part (a), letting $f_n(x) = -\frac{1}{n}$ for $x \in \mathbf{R}$ gives counterexamples to both.
For part (b), the example $f_n = \mathbb{1}_{[n,n+1]}$ from the previous problem does the trick.

5. Suppose that μ is σ-finite and that $f_n \to f$ almost everywhere. Show that there are sets $\{ E_n \}$ such that $E := \bigcup_{n=1}^{\infty} E_n$ is conull[1] and such that $f_n \to f$ uniformly on each E_n. (Compare with #4(b). *Of course*, you should assume that each f_n and f are measurable.)

ANS: Write $X = \bigcup_n X_n$ with each X_n of finite measure and $X_n \subset X_{n+1}$. Then, by Egoroff's Theorem, there is a $E_n \subset X_n$ such that $f_n \to f$ uniformly on E_n and $\mu(X_n \setminus E_n) < \frac{1}{n}$. Thus it suffices to see that $E := \bigcup E_n$ is conull. But

$$\begin{aligned} \mu(X \setminus E) &= \lim_n \mu(X_n \cap \bigcap E_m^c) \\ &\leq \limsup_n \mu(X_n \cap E_n^c) = \limsup_n \mu(X_n \setminus E_n) \\ &= 0. \end{aligned}$$

6. Suppose that $f_n \searrow f$ in L^+. Is it necessarily the case that

$$\int f_n(x)\, d\mu(x) \to \int f(x)\, d\mu(x)?$$

What if $\mu(X) < \infty$? What if $\int f(x)\, d\mu(x) < \infty$? What if $\int f_1(x) < \infty$?

ANS: This works only when some $f_i \in \mathcal{L}^1$. But it is not so easy to find an example when $\mu(X) < \infty$. Several people came up with $f_n(x) = \frac{1}{nx}$ on $(0,1)$. If f_1 has a finite integral, then it is in $\mathcal{L}^1$, and we get convergence by the Dominated Convergence Theorem. Interestingly, if X is compact and the f_i are continuous, then the convergence has to be uniform by Dini's Theorem, and the integrals must converge.

7. Suppose that $f \in L^1(X)$. Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_E |f(x)|\, d\mu(x) < \epsilon$$

provided $\mu(E) < \delta$. (This is easy if f is bounded.)

ANS: If f is bounded, say $|f(x)| \leq M$ for all x, then $\delta = \frac{\epsilon}{M}$ will do. In general, let

$$f_n(x) = \begin{cases} f(x) & \text{if } |f(x)| \leq n, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

[1] We say that E is conull if $\mu(E^c) = 0$.

Then $f_n \to f$ in L^1 by the dominated convergence theorem. Choose n such that $\|f_n - f\|_1 < \frac{\epsilon}{2}$. Then find $\delta > 0$ so that

$$\int_E |f(x)|\, d\mu(x) < \frac{\epsilon}{2}.$$

Then

$$\int_E |f(x)|\, d\mu(x) \leq \int_E |f(x) - f_n(x)|\, d\mu(x) + \int_E |f_n(x)|\, d\mu(x) \leq \|f - f_n\|_1 + \frac{\epsilon}{2} < \epsilon.$$

8. Let f be a function on $[a, \infty)$ such that f is bounded on bounded subsets. Recall that f is improperly Riemann integrable if f is Riemann integrable on each interval $[a, b]$ and

$$\lim_{b\to\infty} \int_a^b f(x)\, dm(x)$$

exits (and is finite). Show that if f is nonnegative ***and Riemann integrable on each*** $[a, b]$ ***with*** $b > a$, then f is improperly Riemann integrable on $[a, \infty)$ if and only if f is Lebesgue integrable on $[a, \infty)$ in which case the value of the Lebesgue integral equals the value of the above limit. What happens when f is not necessarily nonnegative? ("Luke, use the Monotone Convergence Theorem.")