Midterm for Math 103
Due Friday, November 14, 2008

Work on one side of $8\frac{1}{2} \times 11$ inch paper only. Start each problem on a separate page. (This last requirement can be waived for those LaTeX users whose very short and elegant solutions would result in an uncomfortable waste of paper.)

1. Let X be an uncountable set and let $\mathscr{M}$ be the connection of sets E in X such that either E or E^c is at most countable.

(a) Show that $\mathscr{M}$ is a σ-algebra.

(b) Show that

$$\mu(E) := \begin{cases} 1 & \text{if } E \text{ is uncountable, and} \\ 0 & \text{otherwise} \end{cases}$$

is a measure on $(X, \mathscr{M})$

(c) Describe the $\mathscr{M}$-measurable functions $f : X \to \mathbf{R}$ and their integrals.

2. Prove the "missing" results:

(a) Lemma 69: If $\{ f_n \}_{n=1}^{\infty}$ is a sequence of measurable functions which converges to a measurable function f in measure, then every subsequence also converges to f in measure.

(b) Theorem 70: Suppose that $\{ f_n \}_{n=1}^{\infty}$ is a sequence of measurable functions which converges to a measurable function f in measure and that $g \in \mathcal{L}^1(X)$ is such that, for each n, $|f_n(x)| \le g(x)$ for almost all x. Then prove that $f_n \to f$ in $L^1(X)$.

(Part (a) is really very straightforward. It is assigned as more of a hint for the second part than for any other reason.)

3. If $f_n \to f$ pointwise almost everywhere, then must $f_n \to f$ in measure? Does you conclusion change if "almost everywhere" convergence is replace by pointwise convergence everywhere? What if $\mu(X) < \infty$? (Assume that each of f_n and f are measurable.)

4. Counterexamples.

(a) Show that both the Monotone Convergence Theorem and Fatou's Lemma are false without the assumption that the f_n are nonnegative (at least almost everywhere).

(b) Show that Egoroff's Theorem fails if we drop that assumption that $\mu(X) < \infty$.

5. Suppose that μ is σ-finite and that $f_n \to f$ almost everywhere. Show that there are sets $\{ E_n \}$ such that $E := \bigcup_{n=1}^{\infty} E_n$ is conull[1] and such that $f_n \to f$ uniformly on each E_n. (Compare with #4(b). *Of course*, you should assume that each f_n and f are measurable.)

6. Suppose that $f_n \searrow f$ in L^+. Is it necessarily the case that

$$\int f_n(x)\, d\mu(x) \to \int f(x)\, d\mu(x)?$$

What if $\mu(X) < \infty$? What if $\int f(x)\, d\mu(x) < \infty$? What if $\int f_1(x) < \infty$?

7. Suppose that $f \in L^1(X)$. Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that

$$\int_E |f(x)|\, d\mu(x) < \epsilon$$

provided $\mu(E) < \delta$. (This is easy if f is bounded.)

8. Let f be a function on $[a, \infty)$ such that f is bounded on bounded subsets. Recall that f is improperly Riemann integrable if f is Riemann integrable on each interval $[a, b]$ and

$$\lim_{b \to \infty} \int_a^b f(x)\, dm(x)$$

exits (and is finite). Show that if f is nonnegative ***and Riemann integrable on each*** $[a, b]$ ***with*** $b > a$, then f is improperly Riemann integrable on $[a, \infty)$ if and only if f is Lebesgue integrable on $[a, \infty)$ in which case the value of the Lebesgue integral equals the value of the above limit. What happens when f is not necessarily nonnegative? ("Luke, use the Monotone Convergence Theorem.")

[1] We say that E is conull if $\mu(E^c) = 0$.