1. An element m of an R-module M is called a torsion element if there exists a nonzero $r \in R$ with $rm = 0$.

   (a) If R is an integral domain, show that the torsion elements form a submodule $tor(M)$ of M. Also, show that $M/tor(M)$ has no nonzero torsion elements (i.e. it is torsion free).

   (b) Show that if R is not an integral domain, then the torsion elements need not form a submodule.

2. An R-module is called *simple* if it is not the zero module and if it has no proper submodule.

   (a) Prove that any simple module is isomorphic to R/M, where M is a maximal left ideal.

   (b) Prove *Schur's Lemma*: Let $\varphi\colon M \to M'$ be a homomorphism of simple modules. Then either φ is zero, or else it is an isomorphism.

   (c) Prove that $\mathrm{End}_R(M)$ is a division ring if M is simple.

3. Let R be a ring. Consider the ring $M_n(R)$ of $n \times n$ matrices with entries in R.

   (a) Show that any two-sided ideal of $M_n(R)$ is of the form $M_n(I)$, all $n \times n$ matrices with entries in I, for some two-sided ideal I of R.

   (b) Conclude that, if R is a simple ring, meaning that it has no nontrivial proper two-sided ideals, then the ring $M_n(R)$ is also simple.

   (c) If R is a division ring, is the ring $M_n(R)$ simple?

4. For any index set T and R-modules N, M_t, $t \in T$, show that there are group isomorphisms

$$\mathrm{Hom}_R(\bigoplus_{t\in T} M_t, N) \approx \prod_{t\in T} \mathrm{Hom}_R(M_t, N)$$

   and

$$\mathrm{Hom}_R(N, \prod_{t\in T} M_t) \approx \prod_{t\in T} \mathrm{Hom}_R(N, M_t).$$

5. How many group homomorphisms $\mathbb{Z}/12\mathbb{Z} \bigoplus \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}/30\mathbb{Z}$ are there?

6. An object A in a category $\mathcal{C}$ is called an initial object if, for every object X in $\mathcal{C}$, there is a unique morphism $A \to X$. Similarly, an object Z is called a terminal object, if for every object X in $\mathcal{C}$, there is a unique morphism $X \to Z$.

   (a) Prove that initial and terminal objects, if they exist, are unique up to unique isomorphism.

   (b) In the category of rings (with $1 \neq 0$ and morphisms preserving 1), is there an initial object, a terminal object?

(c) Let A and B be objects in a category $\mathcal{C}$. Let $\mathcal{D}_{AB}$ be the category with objects all diagrams in $\mathcal{C}$ of the form

$$A \longrightarrow C \longleftarrow B$$

and morphisms all commuting diagrams of the form

$$\begin{array}{ccccc} A & \longrightarrow & C & \longleftarrow & B \\ & \searrow & \downarrow & \swarrow & \\ & & C' & & \end{array}$$

with the obvious notion of composition. What is the initial object in $\mathcal{D}_{AB}$ if it exists?

7. Show that there is a (noncommutative) ring R with $R \approx R \oplus R$, as R modules. Hint: Consider the endomorphism ring of an infinite-dimensional vector space.

8. A retraction of an R-module map $i\colon M' \to M$ is an R-module map $r\colon M \to M'$ such that $r \circ i = id_{M'}$. Let

$$0 \longrightarrow M' \xrightarrow{i} M \xrightarrow{\pi} M'' \longrightarrow 0$$

be a short exact sequence of R-modules. If i has a retraction, show that $M \approx M' \times M''$. What is the analogous statement in the category of groups?

9. Give a very short proof of the following standard fact in linear algebra: If $T\colon V \to W$ is a linear transformation, then $V \approx \ker T \oplus \operatorname{im} T$.

10. Show that $v = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ extends to a basis $\{v, v_2, \ldots, v_n\}$ of $\mathbb{Z}^n$ if and only if the a_i are coprime, meaning $(a_1) + \cdots + (a_n) = (1)$ as ideals in $\mathbb{Z}$. (Part of this problem can be done quickly using Problem 8.)

11. Let $A = \begin{bmatrix} 4 & 7 & 2 \\ 2 & 4 & 6 \end{bmatrix}$.

   (a) If $\varphi\colon \mathbb{Z}^3 \to \mathbb{Z}^2$ is the homomorphism whose matrix with respect to the standard bases is A, determine the structure of the group $\mathbb{Z}^2/\operatorname{im}\varphi$ as the direct sum of cyclic groups. Find generators (as few as possible) for this quotient group.

   (b) Determine all integer solutions to the system of equations $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

12. Show that if G is a subgroup of the free $\mathbb{Z}$-module $\mathbb{Z}^n$, then there are bases $\{a_1, \ldots, a_k\}$ of G and $\{b_1, \ldots, b_n\}$ of $\mathbb{Z}^n$ such that for each of the basis elements a_i of G, there is a $d_i \in \mathbb{Z}$ with $a_i = d_i b_i$.

13. (a) Show that the group of rationals $\mathbb{Q}^+$ under addition is not a free $\mathbb{Z}$-module, even though it's torsion free.

   (b) Show that the torsion $\mathbb{Z}$-module $\mathbb{Q}^+/\mathbb{Z}^+$ is not an infinite direct sum of cyclic groups.

14. If G is finite abelian group with presentation $0 \longrightarrow \mathbb{Z}^n \xrightarrow{\varphi} \mathbb{Z}^n \longrightarrow G \longrightarrow 0$, show that $|G| = |\det([\varphi])|$, where $[\varphi]$ is the matrix of φ with respect to any bases.

15. Let F be a field and $H \leq F^\times$ a finite subgroup of the multiplicative group of units of F. Show that H is cyclic. (Hint: Use the characterization of cyclic groups in terms of their exponents.)

16. (a) If M and N are finitely generated torsion modules over a PID R, show that

$$\mathrm{Hom}_R(M, N) \approx \bigoplus_p \mathrm{Hom}_R(T_p(M), T_p(N))$$

where the sum is over a finite number of primes p of R.

(b) Describe the structure of the abelian group $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/n\mathbb{Z}, \mathbb{Z}/m\mathbb{Z})$ as a direct sum of cyclic groups (with as few summands as possible).

17. (a) Let V be a finite-dimensional vector space over any field. If $T^2 = \mathrm{Id}$, can T be diagonalized? If so, what are the possible eigenvalues of T?

(b) Same question but assume $T^2 = T$,

(c) $T^2 = 0$.

18. How many $\mathbb{Z}$-bilinear maps are there from $\mathbb{Z} \times \mathbb{Z}$ to G, where G is any finite abelian group? Describe them explicitly.

19. (a) Let I and J be two-sided ideals of a ring R. Show that $\frac{R}{I} \otimes_R \frac{R}{J} \cong \frac{R}{I+J}$.

(b) Show that $\frac{\mathbb{Z}}{m\mathbb{Z}} \otimes_{\mathbb{Z}} \frac{\mathbb{Z}}{n\mathbb{Z}}$ is cyclic. What is its order? Describe a generator explicitly.