# Math 113 – Spring 2005 – Homework #1

1. Fill in the details of the proof of the proposition from lecture that if $T$ is a normal operator on a finite-dimensional complex Hilbert space $\mathcal{H}$, then $\mathcal{H}$ has an orthonormal basis of eigenvectors for $T$.

2. Let $A$ be a normed vector space and let $J$ be a proper subspace of $A$. Also let $\pi : A \to A/J$ be the quotient map from $A$ onto the the quotient vector space $A/J$.

(a) Show that
$$\|\pi(x)\| := \inf_{y\in J} \|x + y\|$$
is a norm on $A/J$. (This norm is called the *quotient norm.*)

(b) Show that for all $\epsilon > 0$, there is a $x \in A$ such that $\|x\| = 1$ and $\|\pi(x)\| \geq 1 - \epsilon$.

(c) Show that $\pi$ has norm 1.

(d) Show that if $A$ is a Banach space, then $A/J$ is a Banach space with respect to the quotient norm.

(e) Show that if in addition, $A$ is a Banach algebra, then $A/J$ is a Banach algebra with respect to the quotient norm.

3. Suppose that $A$ is a Banach space and that $f : [a, b] \to A$ is a continuous function. Recall that a partition $\mathscr{P}$ of $[a, b]$ is simply a finite subset of the form $\{ a = t_0 < t_1 < \cdots < t_n = b \}$. We define $\|\mathscr{P}\| = \max_{1\leq k\leq n} \Delta t_i$, were $\Delta t_i := t_i - t_{i-1}$. If $\zeta \in [a, b]^n = (z_1, z_2, \ldots, z_n)$ is such that $z_i \in [t_{i-1}, t_i]$, then
$$\mathscr{R}(f, \mathscr{P}, \zeta) := \sum_{i=1}^{n} f(z_i)\Delta t_i.$$
We say that $\mathscr{Q}$ is a refinement of $\mathscr{P}$ is $\mathscr{P}$ is a subset of $\mathscr{Q}$.

(a) Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that if $\|\mathscr{P}\| < \delta$ and if $\mathscr{Q}$ is a refinement of $\mathscr{P}$, then
$$\|\mathscr{R}(f, \mathscr{P}, \zeta) - \mathscr{R}(f, \mathscr{Q}, \zeta')\| < \epsilon$$
for any appropriate $\zeta$ and $\zeta'$.

(b) Let $\mathscr{P}_n$ the uniform partition of $[a,b]$ into $2^n$ subintervals, and let $\zeta_n = (t_0, t_1, \ldots, t_{n-1})$. Let

$$a_n = \mathscr{R}(f, \mathscr{P}_n, \zeta_n).$$

Show that $\{ a_n \}$ is Cauchy and define

$$\int_a^b f(t)\,dt := \lim_n a_n.$$

(c) Show that for all $\epsilon > 0$ there is a $\delta > 0$ such that $\|\mathscr{P}\| < \delta$ implies that

$$\|\mathscr{R}(F, \mathscr{P}, \zeta) - \int_a^b f(t)\,dt\| < \epsilon$$

and any appropriate $\zeta$.

(d) Show that if $A$ is a Banach algebra and if $x \in A$, then

$$x \int_a^b f(t)\,dt = \int_a^b x f(t)\,dt \qquad\qquad \int_a^b f(t)\,dt\, x = \int_a^b f(t) x\,dt.$$

(e) Show that if $\Lambda \in A^*$, then

$$\Lambda\Big(\int_a^b f(t)\,dt\Big) = \int_a^b \Lambda\big(f(t)\big)\,dt.$$

In particular, if there is an $x \in A$ such that

$$\Lambda(x) = \int_a^b \Lambda\big(f(t)\big)\,dt \quad \text{for all } \Lambda \in A^*$$

(so that $x$ is the weak integral of $f$), then

$$x = \int_a^b f(t)\,dt.$$