LINEAR OPERATORS ON BANACH SPACES

MATH 113 - SPRING 2015

PROBLEM SET #3

Problem 1. Let E be the space $C([0,1])$ equipped with $\|\cdot\|_\infty$. If f is differentiable, we write $D(f) = f'$.

1. Let F be a closed subspace of E that is included in $C^1([0,1])$.
   (a) Show that $D : F \longrightarrow E$ is Lipschitz.
   (b) Prove that F is finite dimensional.
2. Let $G = (C^1([0,1]), \|\cdot\|_\infty)$.
   (a) Show that $D : G \longrightarrow E$ is closed.
   (b) Is it continuous?

Hints: 1.(a) Study the graph of D. - 1.(b) Study the unit ball of F.

Problem 2. Let E be a normed linear space, F a closed subspace of E and

$$\pi : E \longrightarrow E/F$$

the natural surjection.

1. Let $x \in E$ and $r > 0$. Show that $\pi(B(x,r)) = B(\pi(x), r)$.
2. Let $\mathcal{U}$ be a subset of E/F. Prove that $\mathcal{U}$ is open if and only if $\pi^{-1}(\mathcal{U})$ is open in E.
3. Prove that π is an open map.
4. Show that the Open Mapping Theorem can be deduced from the Bounded Inverse Theorem.

Problem 3 (Bilinear maps)**.** Let E_1, E_2 and F be normed linear spaces and equip $E_1 \times E_2$ with the norm $\|(x,y)\| = \max(\|x\|, \|y\|)$. A map $B : E_1 \times E_2 \longrightarrow F$ is said **bilinear** if all the maps

$$\begin{array}{rcl} \Lambda_x : E_2 & \longrightarrow & F \\ y & \longmapsto & B(x,y) \end{array} \quad \text{and} \quad \begin{array}{rcl} \mathrm{P}_y : E_1 & \longrightarrow & F \\ x & \longmapsto & B(x,y) \end{array}$$

are linear. Moreover, B is said

- **separately continuous** if all the maps Λ_x and P_y are continuous;
- **bounded** if

  $$\|B\| := \sup\left\{\|B(x,y)\| \quad , \quad x \in E_1\ ,\ y \in E_2\ ,\ \|x\| \le 1\ ,\ \|y\| \le 1\right\} < \infty.$$

1. Show that the statements

   (a) B is bounded.

   (b) There exists a constant $C \ge 0$ such that $\|B(x,y)\| \le C\|x\|\|y\|$ for all (x,y) in $E_1 \times E_2$.

   (c) B is continuous.

   (d) B is continuous at $(0,0)$.

   are equivalent and that if they hold, $\|B\|$ is the smallest C satisfying (b).

Recall that the set of bounded linear maps between linear spaces E and F is denoted by $\mathcal{L}(E,F)$. The set of bounded bilinear maps from $E_1 \times E_2$ to F will be denoted by $\mathcal{B}(E_1 \times E_2, F)$.

2. Let E and F be normed linear spaces. Show that the map

   $$\begin{array}{rcl} \beta : \mathcal{L}(E,F) \times E & \longrightarrow & F \\ (T,x) & \longmapsto & T(x) \end{array}$$

   is in $\mathcal{B}\left(\mathcal{L}(E,F) \times E, F\right)$ and that $\|\beta\| \le 1$.

3. Let E, F and G be normed linear spaces. Show that the map

   $$\begin{array}{rcl} \gamma : \mathcal{L}(F,G) \times \mathcal{L}(E,F) & \longrightarrow & \mathcal{L}(E,G) \\ (S,T) & \longmapsto & S \circ T \end{array}$$

   is in $\mathcal{B}\left(\mathcal{L}(F,G) \times \mathcal{L}(E,F), \mathcal{L}(E,G)\right)$ and that $\|\gamma\| \le 1$.

4. Show that $\mathcal{B}(E_1 \times E_2, F)$ equipped with the pointwise operations and $\|\cdot\|$ defined above is a normed linear space.

5. (a) Show that $\mathcal{B}(E_1 \times E_2, F)$ is isometrically isomorphic to $\mathcal{L}\left(E_1, \mathcal{L}(E_2, F)\right)$.

   (b) What can be said of $\mathcal{B}(E_1 \times E_2, F)$ if F is a Banach space?

6. Assume that E_1 and E_2 are Banach spaces. Show that a bilinear map $B : E_1 \times E_2 \longrightarrow F$ is bounded if and only if it is separately continuous.

7. Consider $E = \mathbb{R}[X]$ equipped with the norm $\|P\| = \displaystyle\int_0^1 |\tilde{P}(x)|\, dx$ where $\tilde{P}$ is the function associated with the polynomial P. Show that the bilinear map α defined on $E \times E$ by $\alpha(P, Q) = \displaystyle\int_0^1 \tilde{P}(x)\tilde{Q}(x)\, dx$ is separately continuous but not bounded.