C*-ALGEBRAS

MATH 113 - SPRING 2015

PROBLEM SET #6

Problem 1 (Positivity in C*-algebras)**.** The purpose of this problem is to establish the following result:
Theorem. *Let $\mathcal{A}$ be a unital* C**-algebra. For $a \in \mathcal{A}$, the following statements are equivalent.*

(a) *a is hermitian and $\mathrm{Sp}_{\mathcal{A}}(a) \subset [0, \infty)$*

(b) *There exists b in $\mathcal{A}$ such that $a = b^*b$*

(c) *There exists b hermitian in $\mathcal{A}$ such that $a = b^2$*

An element satisfying **(a)** *is said **positive**and we write $a \geq 0$.*

1. What are the positive elements in $\mathbb{C}$? Verify that the theorem holds in this case.

2. Let $a \in \mathcal{A}$ be hermitian. Prove that there exist positive elements u, v in $\mathcal{A}$ such that $a = u - v$ and $uv = vu = 0$.

3. Let $a \geq 0$ in $\mathcal{A}$ and $n \in \mathbb{N}^*$. Prove the existence of $b \geq 0$ such that $a = b^n$.

4. Verify that **(a)** $\Rightarrow$ **(c)** $\Rightarrow$ **(b)** in the theorem.

5. We want to prove that the elements u, v and b in 2. and 3. are unique. Assume that $a = u' - v'$ with u', v' positive and $u'v' = v'u' = 0$.

   (a) Prove that $P(a) = P(u') + P(-v')$ for any polynomial P.

   (b) Let f be the function defined on $\mathbb{R}$ by $f(t) = \max(t, 0)$. Prove that $u = f(u') + f(-v')$.

   (c) Show that $f(u') = u'$ and $f(-v') = 0$ and conclude.

(d) Use a similar method to prove that the element b in 3. is unique.

Hints: 2.& 3. Functional Calculus, $t \mapsto \max(t,0)$, $t \mapsto \max(-t,0)$, $t \mapsto t^{\frac{1}{n}}$.
5.(a) Start with $P(t) = t^n$.
5.(b) Approach f uniformly on $\mathrm{Sp}(a) \cup \mathrm{Sp}(u') \cup \mathrm{Sp}(v')$ by polynomial functions.

Problem 2 (Non-commutative topology)**.** If $X \stackrel{\varphi}{\to} Y$ is a continuous map between two topological spaces, we denote by $\varphi^\sharp$ the map from $C(Y)$ to $C(X)$ defined by

$$\varphi^\sharp(f) = f \circ \varphi.$$

1. Prove that $X \mapsto C(X)$, $\varphi \mapsto \varphi^\sharp$ is a contravariant functor from the category of compact Hausdorff spaces with continuous maps to the category of commutative unital C*-algebras with $*$-morphisms.

2. What does the Gelfand-Naimark Theorem say about this functor?
   What more can be said?

A map φ between locally compact Hausdorff spaces X and Y is said **proper** if the inverse image of a compact in Y is a compact of X.

3. Show that C_0 is a contravariant functor from the category of locally compact Hausdorff spaces to the category of commutative C*-algebras. Specify the morphisms.

4. Prove that $C_0(X)$ is $*$-isomorphic to $C_0(Y)$ if and only if X and Y are homeomorphic.

5. Assume X compact and $X_0 \subset X$ open.

   (a) Prove that $C_0(X_0)$ is an ideal of $C(X)$.

   (b) Show that all ideals in $C(X)$ are of this form.

6. Complete the following 'dictionary' translating properties of topological spaces in terms of properties of algebras, commutative or not. You may restrict to the case of compact spaces whenever it makes sense.

Spaces	**Algebras**
$\dots$	unital
points	$\dots$
$\dots$	ideals
$\dots$	quotients
$\dots$	$*$-morphism
$\dots$	$*$-isomorphism
disjoint union	$\dots$
connected component	$\dots$

Hints: maximal ideals in $C(X)$ are of the form $\mathcal{J}_{x_0} = \{f \in C(X)\,,\, f(x_0) = 0\}$.
The spectrum of $C_0(X)$ is homeomorphic to X.
A **projection** in a C*-algebra is an element a that satisfies $a^2 = a^* = a$.