FOURIER SERIES

MATH 113 - SPRING 2015

PROBLEM SET #8

Problem 1 (Pointwise and uniform convergence). Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a 2π-periodic function, piecewise continuous, piecewise of class C^1. For $x_0 \in \mathbb{R}$, we denote by $f(x_0^{\pm})$ the one-sided limit $\lim\limits_{x \to x_0^{\pm}} f(x)$ and $\tilde{f}$ is the function defined on $\mathbb{R}$ by

$$\tilde{f}(x) = \frac{f(x^+) + f(x^-)}{2}.$$

The purpose of the problem is to establish the pointwise convergence of the Fourier series of f to $\tilde{f}$, that is, for any $x_0 \in \mathbb{R}$,

$$\sum_{n \in \mathbb{Z}} \hat{f}(n) e^{inx_0} = \tilde{f}(x_0).$$

1. Verify that for any x_0 in $\mathbb{R}$, the map $h \mapsto \dfrac{f(x_0 + h) + f(x_0 - h) - f(x_0^+) - f(x_0^-)}{h}$ is bounded near 0.

First, we consider the case $x_0 = 0$. Denote by $S_N(f)(0)$ the partial sum $\sum\limits_{n=-N}^{N} \hat{f}(n)$.

2. Prove that

$$2\pi S_N(f)(0) = \int_0^{\pi} (f(x) + f(-x)) D_N(x)\, dx,$$

   where $D_N(x)$ is the Dirichlet kernel $\dfrac{\sin(N + \frac{1}{2})x}{\sin \frac{x}{2}}$.

3. Show that $2\pi(S_N(f)(0) - \tilde{f}(0))$ can be written as $\displaystyle\int_0^{\pi} g(x) \sin\left(N + \frac{1}{2}\right) x\, dx$ with g piecewise continuous and bounded near 0.

4. Conclude and extend to the case of arbitrary x_0.

From now on, we assume f continuous and piecewise of class C^1. We denote by φ the function defined on $\mathbb{R}$ by

$$\varphi(x) = \begin{cases} f'(x) & \text{if } f \text{ is differentiable at } x, \\ \dfrac{f'(x^+) + f'(x^-)}{2} & \text{otherwise.} \end{cases}$$

5. Verify the relation $\hat{\varphi}(n) = in\,\hat{f}(n)$ for all $n \in \mathbb{Z}$.

6. Prove that the Fourier series of f converges normally to f.

Hints: 4. Riemann-Lebesgue. Consider $f_{x_0} : x \mapsto f(x+x_0)$. 6. $|ab| \leq \frac{1}{2}(a^2+b^2)$.

Problem 2 (Application to the computation of sums)**.** Let f be the 2π-periodic function on $\mathbb{R}$ defined by $f(x) = 1 - \dfrac{x^2}{\pi^2}$ for all $x \in [-\pi, \pi]$.

1. Compute the Fourier coefficients of f.

2. Deduce the sums of the series $\displaystyle\sum_{n\geq 1} \frac{1}{n^2}$, $\displaystyle\sum_{n\geq 1} \frac{(-1)^n}{n^2}$ and $\displaystyle\sum_{n\geq 1} \frac{1}{n^4}$.

Hints: note that only the real part of $\hat{f}(n)$ is useful. Parseval.

Problem 3 (Not every function is equal to the sum of its Fourier series)**.** Let $\mathcal{C}_{2\pi}$ denote the space of 2π-periodic continuous functions on $\mathbb{R}$, equipped with $\|\cdot\|_\infty$. For $N \in \mathbb{N}$, we define a linear functional φ_N on $\mathcal{C}_{2\pi}$ by

$$\varphi_N(f) = S_N(f)(0) = \sum_{n=-N}^{N} \hat{f}(n)$$

1. Verify that $\mathcal{C}_{2\pi}$ is a Banach space.

2. Prove that $\varphi_N \in \mathcal{C}_{2\pi}^*$ and compute $\|\varphi_N\|$.

3. Show that $\|\varphi_N\| \geq \dfrac{2}{\pi} \displaystyle\int_0^{\frac{(2N+1)\pi}{2}} \left|\frac{\sin u}{u}\right| du$ for any $N \in \mathbb{N}$.

4. Prove the existence of a function in $\mathcal{C}_{2\pi}$ whose Fourier series diverges at 0.

Hints: 2. Consider $f_\varepsilon = \frac{D_N}{|D_N|+\varepsilon}$. 4. Use the Principle of Uniform Boundedness.