Math 113
Homework Assignment Number One
Due Monday, January 22nd

1. Work E.1.2.6 in the text. You should accept as given that, as in E.1.1.8, there is an uncountable well ordered set $(X, \leq)$ such that for all $y \in X$, the set $\{ x \in X : x < y \}$ is countable.[1]

ANS: Let $(X, \leq)$ be an uncountable well-ordered set with the property that for each $y \in X$, the set $\{ x \in X : x \leq y \}$ is countable. Give X the order topology.

First, I claim that X is *not* second countable. Suppose it were and that $\rho = \{ A_n \}_{n=1}^{\infty}$ were a basis for the topology. Thus if V is open in X, we have

$$V = \bigcup \{ A_n \in \rho : A_n \subseteq V \}.$$

Now for each $y \in X$, let $V_y = \{ x \in X : x < y \}$. Note that each V_y is open and countable. Also notice that if $x \in X$ then there is a $y \in X$ such that $x < y$; this is because $\{ z : z \leq x \}$ is countable and X is not. Therefore

$$X = \bigcup_{y \in X} V_y. \tag{1}$$

But the set $S = \{ n \in \mathbf{N} : A_n \subset V_y \text{ for some } y \in X \}$ is certainly countable, and (1) implies that

$$X = \bigcup_{n \in S} A_n. \tag{2}$$

This leads to a contradiction since each A_n is countable (its a subset of a countable set), so (2) implies that X is countable. This proves the claim.

Next I claim that X is first countable. Fix $x \in X$. Since X is well-ordered, $\{ y \in X : x < y \}$ has a least element which I denote x_+. I'll use the notation (y, z) to denote $\{ w \in X : y < w < z \}$. Since $\{ y : y < x \}$ is countable, so is the collection $\rho_x = \{ (y, x_+) : y < x \}$. The latter is clearly a neighborhood base at x. (OK, there's one exception: if 1 denotes the least element of X, then $\{ y : y < 1 \}$ is empty and, therefore, so is ρ_1. But in that case $\{ \{ 1 \} \}$ is a neighborhood base at 1. Notice that if x has a predecessor, i.e., an element x_- such that $(x_-)_+ = x$, then $\{ \{ x \} \}$ is a neighborhoodbase at x. But not every point will have a predecessor.)

2. Work E.1.2.9 in the text.

ANS: Let (X, d) be a metric space. Fix $x \in X$. Let $B_\epsilon(x) = \{ y \in X : d(x, y) < \epsilon \}$. I claim $\mathcal{N} = \{ B_{\frac{1}{n}}(x) \}_{n=1}^{\infty}$ is a neighborhood base at x. Let V be a neighborhood of x. Then there is an $\epsilon > 0$ such that $x \in B_\epsilon(x) \subseteq V$. Choose n such that $\frac{1}{n} < \epsilon$. Now $x \in B_{\frac{1}{n}}(x) \subseteq B_\epsilon(x) \subseteq V$. This proves the claim.

[1] It is not part of this problem, but in the order topology on X, closed intervals, $[x, y] := \{ z : x \leq z \leq y \}$, are compact and X is locally compact.

Next suppose that (X, d) is second countable; let $\{ A_n \}_{n=1}^\infty$ be a basis for the topology. For each $n \geq 1$, choose any $x_n \in A_n$. I claim that $D = \{ x_n \}_{n=1}^\infty$ is dense in X, and hence, that X is separable. But if V is a nonempty open subset of X, say $x \in V$, then there is a n such that $x \in A_n \subset V$. Thus, $D \cap V \neq \emptyset$. The claim follows.

Now suppose that (X, d) is separable; suppose that $D = \{ x_n \}_{n=1}^\infty$ is dense. Let $\rho = \{ B_{\frac{1}{m}}(x_n) : n, m \geq 1 \}$. I will show that ρ is a basis and then it follows by definition that X must be second countable. Let V be open in X and suppose that $x \in V$. Then there exists $\epsilon > 0$ such that $B_\epsilon(x) \subseteq V$. Choose m so that $\frac{1}{m} < \frac{\epsilon}{2}$. Since D is dense, there is a x_n such that $d(x_n, x) < \frac{1}{m}$. Then if $y \in B_{\frac{1}{m}}(x_n)$, we have $d(y, x) \leq d(y, x_n) + d(x_n, x) < \frac{1}{m} + \frac{1}{m} < \epsilon$. Thus,

$$x \in B_{\frac{1}{m}}(x_n) \subseteq B_\epsilon(x) \subseteq V.$$

This establishes the claim.

3. Work E.2.1.1 in the text.

ANS: The interesting part of this problem is to show that "absolute convergence implies convergence" implies completeness.

Suppose that $\{ x_n \}_{n=1}^\infty$ is a Cauchy sequence. We accept that it suffices to show that $\{ x_n \}_{n=1}^\infty$ has a convergent subsequence. Using the definition of Cauchy sequence, an induction argument shows that there is a subsequence $\{ x_{n_k} \}_{k=1}^\infty$ such that

$$\|x_{n_{k+1}} - x_{n_k}\| < \frac{1}{2^k}.$$

Let $y_1 := x_{n_1}$, and if $k \geq 2$, let $y_k := x_{n_{k+1}} - x_{n_k}$. Then $\sum y_k$ is absolutely convergent. By assumption, there is a $x \in X$ such that

$$x = \lim_k \sum_{i=1}^k y_i = \lim_k x_{n_k}.$$

That's what we wanted.

4. Work E.2.1.4 in the text.

5. Let X be a normed vector space and let $B = \{ x \in X : \|x\| \leq 1 \}$ be the *unit ball*. Show that if B is compact, then X is finite dimensional.[2] Since this is E.2.1.3 in the text, I was embarrassed not to be able to give a "qucik" proof. You can either follow my steps below, or provide a better proof yourself.

[2]It is easy to go from here to showing that any normed vector space that is locally compact is necessarily finite dimensional.

(a) Let $V = \{ x \in X : \|x\| < 1 \}$ be the open unit ball. Show that there is a finite set $\{ x_1, \dots, x_n \} \subset X$ such that

$$B \subset \bigcup_{i=1}^{n} x_i + \frac{1}{2}V.$$

(b) Let $Y = \operatorname{span}\{ x_1, \dots, x_n \}$ and conclude that

$$V \subset Y + \frac{1}{2}V$$

(c) Let $q : X \to X/Y$ be the quotient map. Show that $W := q(V)$ is open.

(d) Show that $2W \subset W$.

(e) Deduce that $X/Y = \{ 0 \}$ so that $X = Y$ is finite dimensional as claimed.

ANS: Actually, Chor's solution was a bit cleaner than mine. (Well, his idea was cleaner, his execution left a bit to the imagination.)

Once you have $V \subset Y + \frac{1}{2}V$, we have $V \subset Y + \frac{1}{2}(Y + \frac{1}{2}V) = Y + \frac{1}{4}V$. Iterating,

$$V \subset Z := \bigcap_{n} (Y + \frac{1}{2^n}V).$$

Clearly, $Y \subset Z$. But it $z \in Z$, then there are $y_n \in Y$ such that $\|z - y_n\| < \frac{1}{2^n}$. But then $y_n \to z$, and since Y is closed, $z \in Y$. Thus $Z = Y$, and we have

$$V \subset Y.$$

This implies that $Y = X$. Therefore X is finite dimensional (since Y is).

6. Define two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a vector space V to be equivalent in they determine the same topology on V. Prove that $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if and only if there are nonzero positive constants c and d such that

$$c\|v\|_1 \le \|v\|_2 \le d\|v\|_1 \quad \text{for all } v \in V.$$

ANS: This is similar to a result proved in lecture. Let T be the identity map from $(V, \|\cdot\|_1) \to (V, \|\cdot\|_2)$. If the norms are equivalent, then T is continuous and therefore bounded. Hence there is a $d \ge 0$ such that $\|x\|_2 = \|Tx\|_2 \le d\|x\|_1$ for all x. Since $\|\cdot\|_2$ is a norm, $d > 0$. Since T^{-1} is also continuous, there is a $c > 0$ such that $\|x\|_1 \le \frac{1}{c}\|x\|_2$ for all x. This is what we wanted to prove.

If the norm inequalities hold, then T and T^{-1} are bounded maps — therefore they are continuous with respect to the topologies induced by the norms. In other words, the identity map is a homeomorphism. Therefore the topologies coincide.

7. Work E.2.1.11 in the text. CORRECTION: **Assume $\mathfrak{X}$ and $\mathfrak{Y}$ are Banach spaces.**

ANS: We need X and Y to be Banach spaces.

Let $X_0 = \operatorname{span}\{\, x_j : j \in J \,\}$. Suppose that there is a $\alpha \in \mathbf{R}$ such that for every finite set $\lambda \subset J$, we have

$$\Big\|\sum_{j\in\lambda} \alpha_j y_j\Big\| \le \alpha \Big\|\sum_{j\in\lambda} \alpha_j x_j\Big\|. \tag{3}$$

Since $\sum_{j\in\lambda} \alpha_j x_j$ is a typical element of X_0, we want to define $T_0 : X_0 \to Y$ by

$$T_0\Big(\sum_{j\in\lambda} \alpha_j x_j\Big) = \sum_{j\in\lambda} \alpha_j y_j. \tag{4}$$

But to do this properly, that is, to see that T_0 is "well-defined", we need to see that if

$$\sum_{j\in\lambda} \alpha_j x_j = \sum_{i\in\lambda'} \beta_i x_i,$$

then

$$\sum_{j\in\lambda} \alpha_j y_j = \sum_{i\in\lambda'} \beta_i y_i.$$

To see this, it suffices to see that $\sum_{j\in\lambda} \alpha_j x_j = 0$, then $\sum_{j\in\lambda} \alpha_j y_j = 0$. But this follows easily from (3). Therefore we get a well-defined operator $T_0 : X_0 \to Y$, and (3) implies that T_0 is bounded by α. Then Proposition 2.1.11 implies that there is a $T : X \to Y$ extending T_0. Clearly $T(x_j) = y_j$. The other direction is straightforward.

8. (After Monday's lecture): Suppose that X is a locally compact Hausdorff space. Show that $C_0(X)$ is closed in $C^b(X)$ and that $C_c(X)$ is dense in $C_0(X)$.

ANS: Suppose that $f_n \to f$ with each $f_n \in C_0(X)$. We want to see that $f \in C_0(X)$. Fix $\epsilon > 0$. It suffices to see that

$$\{\, x : |f(x)| \ge \epsilon \,\}$$

is compact. Since it is clearly closed, it is enough to see that it is contained in a compact set. But we can choose n such that $\|f - f_n\|_\infty < \epsilon/2$. Then

$$\{\, x : |f(x)| \ge \epsilon \,\} \subset \{\, x : |f_n(x)| \ge \epsilon/2 \,\}.$$

Since the latter is compact, this suffices to show that $C_0(X)$ is closed in $C^b(X)$.

To see that $C_c(X)$ is dense in $C_0(X)$, let $f \in C_0(X)$. Fix $\epsilon > 0$. Then by assumption,

$$K := \{\, x : |f(x)| \ge \epsilon \,\}$$

is compact. By the version of Urysohn's lemma we proved in lecture, there is a continuous function $\phi : X \to [0,1]$ with compact support such that $\phi(x) = 1$ for all $x \in K$. Since $\phi \in C_c(X)$, so is the pointwise product ϕf. But

$$\|f - \phi f\|_\infty \le \epsilon$$

(since $|f(x) - \phi(x)f(x)| = (1-\phi(x))|f(x)| \le |f(x)|$). This suffices.

Remark. *I thought E.2.1.6, E.2.1.8, E.2.1.9 and E.2.1.10 all illustrated some interesting examples of Banach spaces, but I couldn't bear the thought of more to grade.*