Homework Assignment #4
Due Wednesday, February 28$^{\text{th}}$

1. In this problem, X will be a *separable* Banach space. Let B^* be the closed unit ball in X^*. We want to work out a solution to E 2.5.3 in the text. Work out your own solution, or follow the guidelines below.

(a) Show that a subset of separable metric space is separable so that we can find a countable dense subset $\{ d_k \}_{k=1}^{\infty}$ of the unit sphere $S = \{ x \in X : \|x\| = 1 \}$ in X. (Hint: a separable metric space is second countable.)

(b) For each k, show that $m_k(\varphi) := |\varphi(d_k)|$ is a seminorm on X^* such that $m_k(\varphi) \leq 1$ on B^*.

(c) Show that a net $\{ \varphi_j \}$ in B^* converges to $\varphi \in B^*$ in the weak-$*$ topology if and only if $m_k(\varphi_j - \varphi) \to 0$ for all k.

(d) For each $\varphi, \psi \in B^*$, define

$$\rho(\varphi, \psi) := \sum_{n=1}^{\infty} \frac{m_n(\varphi - \psi)}{2^n}.$$

Show that ρ is a metric on B^*.

(e) Show that a net $\{ \varphi_j \}$ in B^* converges to $\varphi \in B^*$ in the weak-$*$ topology if and only if $\rho(\varphi_j, \varphi) \to 0$. Conclude that the topology induced by ρ on B^* is the weak-$*$ topology; that is, conclude that the weak-$*$ topology on B^* is metrizable.

(f) Conclude that X^* is separable in the weak-$*$ topology. (As Pedersen points out, a compact metric space is totally bounded and therefore separable.)

2. Work E 2.5.6, but use the hint from the "revised edition" of the text.

3. Suppose that H is an inner product space. Show that $|(x \mid y)| = \|x\|\|y\|$ if and only if either $x = \alpha y$ or $y = \alpha x$ for some $\alpha \in \mathbf{F}$.

4. Suppose that W is a nontrivial subspace of a Hilbert space H. Define $P : H \to W$ by $P(h) = w$, where w is the closest element in W to h. (Alternatively, $P(h) = w$ where $h = w + w^{\perp}$ with $w \in W$ and $w^{\perp} \in W^{\perp}$.)

(a) Show that P is a bounded linear map with $\|P\| = 1$.

(b) If Y is a subspace of H and if $y \in Y$ is in the kernel of P, then $y \in W^{\perp}$.

(c) Work problem E 3.1.7 in the text.

5. Work problem E 3.1.9 in the text. (Remark: in problem 1 implies that H is separable in the weak topology. Here we also see that, despite this, it fails to be either second countable or even first countable in the weak topology.)

6. Work problem E 3.1.11 in the text.