Homework Assignment #4
Due Wednesday, February 17th

INSTRUCTIONS: As usual, for the "true/false" questions, just circle the correct answer. No justifications are required, but don't guess. You score is based on #right minus #wrong.

1. **TRUE or FALSE**: If X is a normed vector space and if φ is a weakly-continuous linear functional on X, then φ is norm continuous. (Recall that the weak topology on X is the weak topology induced by the pairing with X^*.)

2. **TRUE or FALSE**: If X is a normed vector space and if φ is a norm continuous linear functional on X, then φ is weakly continuous.

3. **TRUE or FALSE**: If X is a normed vector space and ψ is a functional on X^* which is continuous in the weak-$*$ topology, then ψ is norm continuous.

4. **TRUE or FALSE**: If X is a normed vector space and ψ is a functional on X^* which is continuous in the norm topology, then ψ is weak-$*$ continuous.

5. **TRUE or FALSE**: If X is a reflexive Banach space, then the unit ball in X is weakly compact.

6. **TRUE or FALSE**: Let X be a normed vector space and let Y be a dense subspace of X^*. Then the weak topology on X induced by the linear functionals in Y coincides with the weak topology on X (induced by all the functionals in X^*).

7. Recall that a set C in a vector space X is called convex if $x, y \in C$ and $\lambda \in [0,1]$ implies that $\lambda x + (1-\lambda)y \in C$.

(a) Suppose that $x_1, \dots, x_n \in X$. If $\lambda_i \geq 0$ and $\sum_{i=1}^n \lambda_i = 1$, then $\sum_{i=1}^n \lambda_i x_i$ is called a convex combination of the x_i. Show if C is convex, then any convex combination of elements from C belongs to C.

(b) Show that if C is a convex subset of a topological vector space X, then its closure, $\overline{C}$ is also convex.

(c) Work problem E 2.4.1 in the text.

8. Let $\{ F_j \}_{j \in J}$ be a collection of nonempty closed subsets in a compact space X which is totally ordered by reverse containment.[1] Then

$$\bigcap_{j \in J} F_j \neq \emptyset.$$

9. Suppose that X is a compact topological space and that $f : X \to \mathbf{R}$ is continuous. Show that f attains its maximum and minimum on X; that is, show that there are points $y, z \in X$ such that

$$f(y) \leq f(x) \leq f(z) \quad \text{for all } x \in X.$$

(Hint: use Theorem 1.6.2(v).)

10. Work E 2.4.5.

11. Work E 2.4.6.

12. Suppose that m is a Minkowski functional on a topological vector space X. Let $C := \{ x \in X : m(x) < 1 \}$. Suppose that $\alpha C = C$ for all $\alpha \in \mathbf{F}$ with $|\alpha| = 1$. Show that $m(x) \geq 0$ for all $x \in X$.

13. Work E 2.4.16 in the "Revised Printing" of the text. If you don't have access to the revised printing, email me, and I'll send you a pdf of the relevant problem page.[2] (You may want to use question 12.)

[1] Recall that "ordered by reverse containment" simply means that $F_j \geq F_{j'}$ if and only if $F_j \subset F_{j'}$.

[2] This problem is key to E 2.4.17 which implies that *every* locally convex topological vector space topology arises from a family of seminorms. You can consider E 2.4.17 an optional "extra-credit" problem. (That means you are welcome to discuss your write-up with me, but don't turn it in.)