THE JORDAN-VON NEUMANN THEOREM

Proposition 1. *Suppose that X is a complex*[1] *Banach space whose norm satisfies the parallelogram law:*

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2. \tag{1}$$

Then X is a Hilbert space. More precisely, the form

$$(x \mid y) := \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2 \tag{2}$$

is an inner product on X such that $(x \mid x) = \|x\|^2$.

We proceed with a sequence of lemmas.

Lemma 2. *For each $x \in X$, $(x \mid x) = \|x\|^2$.*

Proof. Using the homogeneity of $\|\cdot\|$:

$$4(x \mid x) = \|2x\|^2 + i\|(1+i)x\|^2 - 0 - i\|(1-i)x\|^2 = 4\|x\|^2. \quad \square$$

Corollary 3. *For all $x \in X$, $(x \mid x) \geq 0$ and $(x \mid x) = 0$ only if $x = 0$.*

Lemma 4. *For all $x, y \in X$, we have $(y \mid x) = \overline{(x \mid y)}$.*

Proof. Again, using the homogeneity of $\|\cdot\|$:

$$\begin{aligned} 4(x \mid y) &= \|x+y\| + i\|x+iy\| - \|x-y\| - i\|x-iy\| \\ &= \|x+y\| + i\|y-ix\| - \|y-x\| - i\|y+ix\| \\ &= 4\overline{(y \mid x)} \quad \square \end{aligned}$$

The next lemma is the key step. Of course, it was suggested by the exercise in Knapp's book. Nevertheless, it still found it tricky to work out. I am confident that there is a better way.

Lemma 5. *For all $x, y, z \in X$, we have*

$$\|x+y+z\|^2 = \|x+y\|^2 + \|x+z\|^2 + \|y+z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2. \tag{3}$$

This proof was taken from [1, Chap. XII §7 Exercises 19–24.]

[1]This proof can be easily modified for a real Banach space: simply replace (2) with

$$(x \mid y) := \frac{1}{4}\big(\|x+y\|^2 - \|x-y\|^2\big). \tag{†}$$

Proof. We make repeated use of the parallelogram law (to the indicated term):

$$\begin{aligned}
\|x+y+z\|^2 &= 2\|x+y\|^2 + 2\|z\|^2 - \|x+y-z\|^2 \\
&= \|x+y\|^2 + \|z\|^2 + \underbrace{\|x+y\|^2 + \|z\|^2} - \|x+y-z\|^2 \\
&= \|x+y\|^2 + \|z\|^2 + \frac{1}{2}\underbrace{\|x+y+z\|^2} - \frac{1}{2}\|x+y-z\|^2 \\
&= \|x+y\|^2 + \|z\|^2 + \frac{1}{2}\big(2\|x+z\|^2 + 2\|y\|^2 - \|x-y+z\|^2\big) - \frac{1}{2}\|x+y-z\|^2 \\
&= \|x+y\|^2 + \|x+z\|^2 + \|y\|^2 + \|z\|^2 - \frac{1}{2}\big(\underbrace{\|x-y+z\|^2 + \|x+y-z\|^2}\big) \\
&= \|x+y\|^2 + \|x+z\|^2 + \underbrace{\|y\|^2 + \|z\|^2} - \|x\|^2 - \|z-y\|^2 \\
&= \|x+y\|^2 + \|x+z\|^2 + \frac{1}{2}\|y+z\|^2 + \frac{1}{2}\|z-y\|^2 - \|x\|^2 - \|z-y\|^2 \\
&= \|x+y\|^2 + \|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \|x\|^2 - \frac{1}{2}\underbrace{\|z-y\|^2} \\
&= \|x+y\|^2 + \|x+z\|^2 + \frac{1}{2}\|y+z\|^2 - \|x\|^2 - \frac{1}{2}\big(2\|y\|^2 + 2\|z\|^2 - \|y+z\|^2\big) \\
&= \|x+y\|^2 + \|x+z\|^2 + nsy + z\|x\|^2 - \|y\|^2 - \|z\|^2. \quad \square
\end{aligned}$$

Having successfully dealt with that messy computation, the fact that the potential inner product preserves sums is easy.

Lemma 6. *For all $x, y, z \in X$, we have $(x+y \mid x) = (x \mid z) + (y \mid z)$.*

Proof. The essential observation is that $\sum_{k=0}^{3} i^k = 1 + i - 1 - i = 0$. Then using Lemma 5, we have

$$\begin{aligned}
4(x+y \mid x) &= \sum_{k=0}^{3} i^k \|x+y+i^k z\|^2 \\
&= \sum_{k=0}^{3} i^k \big(\|x+y\|^2 + \|x+i^k z\|^2 + \|y+i^k z\|^2 - \|x\|^2 - \|y\|^2 - \|z\|^2\big) \\
&= 0 + \sum_{k=0}^{3} \|x+i^k z\|^2 + \sum_{k=0}^{3} \|y+i^k z\|^2 + 0 + 0 + 0 \\
&= 4(x \mid z) + 4(y \mid z). \quad \square
\end{aligned}$$

It would seem now that we are all but done. But showing that the potential inner product respects scalar multiplication is not so easy. We have to work with the complex rationals $\mathbf{D} = \mathbf{Q} + i\mathbf{Q}$.

Lemma 7. *Suppose that $r \in \mathbf{D}$ and that $x, y \in X$, then $(rx \mid y) = r(x \mid y)$.*

Proof. It follows immediately from Lemma 6, that for all $n \in \mathbf{N}$, we have $(nx \mid y) = n(x \mid y)$. It is then a simple matter to see that $(rx \mid y) = r(x \mid y)$ for all

$r \in \mathbf{Q}$. On the other hand,

$$\begin{aligned} 4(ix \mid y) &= \sum_{k=0}^{3} i^k \|ix + i^k y\|^2 \\ &= \|ix+y\|^2 + i\|ix+iy\|^2 - \|ix-y\|^2 - i\|ix-iy\|^2 \end{aligned}$$

which, by homogeneity, is

$$\begin{aligned} &= i\big(-i\|x-iy\|^2 + \|x+y\|^2 - i\|x-iy\|^2 - \|x-y\|^2\big) \\ &= 4i(x \mid y). \end{aligned}$$

combining this with the first part of the proof and Lemma 6 gives the result. □

Upgrading from $r \in \mathbf{Q}$ to $c \in \mathbf{C}$ requires that we prove that the Cauchy Schwarz inequality holds using only the tools at our disposal so far. Fortunately, the usual proofs works just fine.

Lemma 8. *For all $x, y \in X$, $|(x \mid y)| \leq \|x\|\|y\|$.*

Proof. We can assume that $y \neq 0$. Note that for all $r \in \mathbf{D}$,

$$\begin{aligned} 0 \leq \|x - ry\|^2 &= (x - ry \mid x - ry) \\ &= \|x\|^2 - 2\operatorname{Re}\bar{r}(x \mid y) + |r|^2\|y\|^2. \end{aligned}$$

But we can find a sequence or rationals $r_n \to \frac{(x|y)}{\|y\|^2}$. Then taking limits in the above,

$$0 \leq \|x\|^2 - 2\frac{|(x \mid y)|}{\|y\|^2} + \frac{|(x \mid y)|}{\|y\|^2},$$

and the result follows. □

Lemma 9. *For all $c \in \mathbf{C}$, we have $(cx \mid y) = c(x \mid y)$.*

Proof. Let $r_n \to c$ with each $r_n \in \mathbf{D}$. Then using Lemma 8, we have

$$|(r_n x \mid y) - (cx \mid y)| \leq |r_n - c|\|x\|\|y\|,$$

and $(r_n x \mid y) \to (cx \mid y)$. But by Lemma 7, $(r_n x \mid y) = r_n(x \mid y) \to c(x \mid y)$. This suffices. □

Proof of Proposition 1. It follows from Lemma 6 and Lemma 9 that $(\cdot \mid \cdot)$ is linear in its first variable. By Lemma 4 it is conjugate linear in its second variable, and both positive and definite by Corollary 3. Therefore $(\cdot \mid \cdot)$ is an inner product. It defines the original, and therefore complete, norm on X by Lemma 2. The Proposition follows. □

References

[1] Anthony W. Knapp, *Basic real analysis*, Cornerstones, Birkhäuser Boston Inc., Boston, MA, 2005. Along with a companion volume *Advanced real analysis*. MR **MR2155259 (2006c:**26002)