MATH 116 WORKSHEET : Eigenvalue bounds for lin. ops.

11/13/08
Barnett

Thm: Let $\mathcal{H}$ be Hilbert space, A an operator with domain $\mathcal{D}(A) \subset \mathcal{H}$, with complete set of eigenfunctions ϕ_j and eigenvalues E_j (discrete).
Let $0 \neq u \in \mathcal{D}(A)$, $E \in \mathbb{R}$, define the residual $r := Au - Eu$
Then $\exists j$ s.t. $|E - E_j| \leq \dfrac{\|r\|}{\|u\|}$ ($\|\cdot\|$ is 2-norm).

Prove this in following steps: since $\{\phi_j\}$ complete o.n.b. write $u = \sum c_i \phi_i$

compute $\|u\|^2$ in terms of c_i : [hint: Parseval]

Compute $\|r\|^2$ in terms of c_i :

Bound $\|u\|^2$ by $\|r\|^2$ times the largest term inside the sum: [hint: put $\frac{(E-E_i)^2}{(E-E_i)^2}$ inside $\|u\|^2$].

Square-root your inequality: QED.

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— SOLUTIONS —

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Prove this in following steps : since $\{\phi_j\}$ complete o.n.b. write $u = \sum c_i \phi_i$

compute $\|u\|^2$ in terms of c_i : $(u,u) = \left(\sum_i c_i \phi_i, \sum_j c_j \phi_j\right)$ [hint: Parseval]

$$= \sum_{ij} \bar{c}_i c_j \underbrace{(\phi_i, \phi_j)}_{\delta_{ij}} = \sum |c_i|^2$$ this is Parseval.

Compute $\|r\|^2$ in terms of c_i :

$$r = (A - E) \sum c_i \phi_i = \sum c_i (E_i - E) \phi_i$$

$$\|r\|^2 = \sum |c_i|^2 (E_i - E)^2$$

Bound $\|u\|^2$ by $\|r\|^2$ times the largest term inside the sum: [hint: put $\frac{(E-E_i)^2}{(E-E_i)^2}$ inside $\|u\|^2$]

$$\|r\|^2 \geq \min_i (E_i - E)^2 \underbrace{\sum |c_i|^2}_{\|u\|^2}$$

Square-root your inequality : QED.

$$\min_i |E - E_i| \leq \frac{\|r\|}{\|u\|}.$$

see Thm 1, G. Still, Numerische Matematik (1988) 54, 201-223.