Conditioning & Stability (§12) (Lec3)

property of ~~the~~ a math problem — property of alg. used to solve it.

Problem is map $f : X \to Y$ (problem input space → space of solns). eg. $f(x)$ could { return $2x$; return vector of roots of polynomial given x = vec. of coeffs of poly. }

a prob. Well-conditioned if infinitesimal perturbation δx (one symb.) in x causes 'small' pert $\delta f := f(x+\delta x) - f(x)$ in solns.

Abs. condition # $\hat{\kappa} = \hat{\kappa}(x) := \lim_{\delta \to 0} \sup_{\|\delta x\| \le \delta} \frac{\|\delta f\|}{\|\delta x\|}$ abbrev. $= \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|}$ (infinitesimal)

let's use 2-norms

express in derivatives: Jacobian $J(x)$ elements $\frac{\partial f_i}{\partial x_j}(x)$. As $\|\delta x\| \to 0$ we have $\delta f \approx J(x)\,\delta x$

so $\hat{\kappa}(x) = \|J(x)\|_2$

more useful is Rel. cond # $\kappa := \sup_{\delta x} \frac{\|\delta f\| / \|f\|}{\|\delta x\| / \|x\|} = \frac{\|J(x)\|_2}{\|f\| / \|x\|}$ important since computer introduces rel. errors.

$\kappa < 10^3$ 'well-cond'

$\kappa \gg 10^3$ ill-cond

Basic operations

- $f(x) = x/2$ $\quad J = f' = 1/2$ so $\kappa = 1$.
- $f(x) = x^\alpha$ $\quad J = f' = \alpha x^{\alpha-1}$ so $\kappa = \frac{|\alpha x^{\alpha-1}|}{x^\alpha / x} = |\alpha| \ll 10^3$ for reasonable powers
- $f(x_1, x_2) = x_1 - x_2$ $\quad J = [1 \; -1]$ $\quad \|J\|_2 = \sqrt{2}$ $\quad \kappa = \frac{\sqrt{2}\sqrt{x_1^2 + x_2^2}}{|x_1 - x_2|} \to \infty$ as $x_1 \to x_2 \ne 0$
- $f(x) = \sin x$ for $x \approx 10^{100}$: $\|J\| \le 1$ but $\kappa = \frac{\|J\|_2}{|\sin x| / |x|} \gtrsim |x| =$ huge!

rel. cond. # depends here on argument x.

- finding poly roots is ill-cond.
- eigvals of non-symm. matrices eg. $A = \begin{bmatrix} 1 & 10^3 \\ & 1 \end{bmatrix}$ vs. $\begin{bmatrix} 1 & 10^3 \\ 10^{-3} & 1 \end{bmatrix}$ $\quad \|\delta x\| = 10^{-3}$, $\|\delta f\| \approx 1$

$\lambda = \{1, 1\}$ $\qquad \lambda = \{0, 2\}$

but $\hat{\kappa} = 1$ if $A^* = A$.

Mat-vec. mult. $f(x) = Ax$, $A \in \mathbb{C}^{m \times m}$ $\quad \kappa = \|A\| \cdot \frac{\|x\|}{\|Ax\|}$ ($J = A$) → if A sq., nonsing, $\le \|A^{-1}\|$ why?

$\|A^{-1}Ax\| \le \|A^{-1}\| \cdot \|Ax\|$ (defn of matrix 2-norm..)

so $\kappa \le \|A\| \|A^{-1}\|$ $\qquad$ equality if $x = v_m$.

If A nonsing then solving $Ax = b$ is $f(b) = A^{-1}b$. Replace A by A^{-1} in above, get again $\kappa \le \|A^{-1}\| \|A\|$ equality if $x = u_1$

We call $\|A\| \|A^{-1}\| =: \kappa(A)$ 'cond # of matrix A'. $= \frac{\sigma_1}{\sigma_m}$ = eccentricity of hyperellipse.

(PTO)

What if A is perturbed instead of b?

$$(A + \delta A)(x + \delta x) = b$$

$$\cancel{Ax} + \delta A x + A \delta x + \delta A \delta x = \cancel{b}$$

v. small.

so $\|\delta x\| \le \|A^{-1}\| \, \|\delta A\| \, \|x\|$.

Thm: 12.2
$$\frac{\|\delta x\| / \|x\|}{\|\delta A\| / \|A\|} \le \|A^{-1}\| \|A\| = \kappa(A)$$

OPENING:

since A, b stored to eg. 16 digits, expect to lose $\log_{10} \kappa(A)$ digits in acc. of x. [redacted]

Now more.

(Lec 3)

Floating pt.

$x \in \mathbb{R}$ digital rep: finite # bits $\Rightarrow$ finite subset F of $\mathbb{R}$ $\Rightarrow$ must be $\left\{ \begin{array}{l} \text{lowest \& highest} \to 10^{\pm 308} \text{ in IEEE double prec.} \\ \text{gaps} \end{array} \right.$

e.g. $[1,2]$ is rep. by set $\{1, \underbrace{1+2^{-52}}, 1+2\cdot 2^{-52}, \dots \quad 2\}$

$(2,4]$ is twice these

relative gap 2.2×10^{-16} never exceeded. : however poor alg (unstable) can cause this to dominate.

formally, base $\beta = 2$

precision $t = 53$

set $F = \{0, \pm \frac{m}{\beta^t}\beta^e, \underbrace{\pm \text{Inf}, \text{NaN}}_{\text{special codes, rather than members of } \mathbb{R}.}\}$

$\beta^{t-1} \le m < \beta^t$ mantissa (so m/β^t in $[\frac{1}{\beta}, 1)$) $\quad e \in \mathbb{Z}$ exponent (we ignore over/under-flow: that there's a largest $|e|$)

NB $F = \beta F$ self-similar

$\varepsilon_{mach} = \frac{1}{2}\beta^{1-t}$ is largest relative gap, ie $\forall x \in \mathbb{R} \ \exists x' \in F$ s.t. $|x - x'| \le \varepsilon_{mach} x$

let such an x' be called $fl(x)$

$\boxed{\text{Then } \forall x \in \mathbb{R} \ \exists |\varepsilon| \le \varepsilon_{mach} \text{ s.t. } fl(x) = (1+\varepsilon)x}$

limited relative err.

IEEE double prec, $\varepsilon_{mach} = 2^{-53} \approx 1.1\times 10^{-16}$

Arithmetic let $\oplus \ominus \otimes \oslash$ be analogs of $+ - \div \times$ except done by machine

it could hold that $x \circledast y = fl(x * y)$

but all we need is weaker prop.

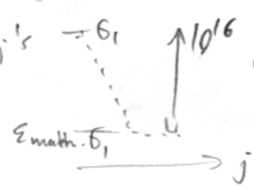
Fund. Axiom of floating pt: $\quad \forall x, y \in F \ \exists \varepsilon$ with $|\varepsilon| \le \varepsilon_{mach}$ s.t. $x \circledast y = (1+\varepsilon)(x * y)$

ie rel. err. at most ε_{mach}.

turns out for $\mathbb{C}$ arith, ε_{mach} can be replaced by $2^{5/2}\varepsilon_{mach}$, similar.

Next time: combine conditioning & errors to get stability.

SVD of vander(50): check its σ_j's



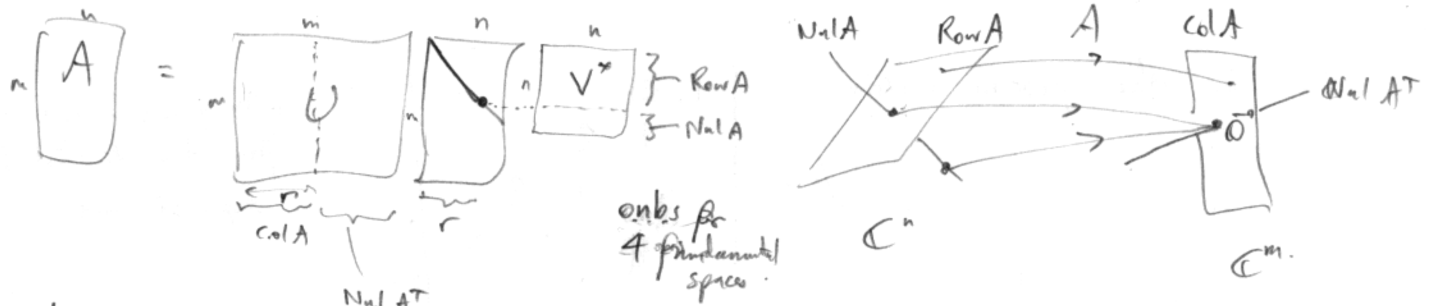
$\leftarrow$ computing σ_j's is stable but best error you can do is $\sigma_1 \cdot \varepsilon_{mach}$.

$\Rightarrow$ when $\sigma_m < \sigma_1 \varepsilon_{mach}$, true rank cannot be determined.

Lec 3 M116 • conclude w/ bkw st. of A\b via SVD. ① 8/2/08
or via Gauss elim w/ partial pivot. for m=n.
or QR (Thm. 16-2).

review:

SVD



$r = \text{rank}A = \dim \text{Col}A = \#\{j : \sigma_j > 0\}$ mathematical object

real world, numerical rank $r_\varepsilon = \#\{j : \sigma_j > \varepsilon\}$ where $\varepsilon > 0$ tolerance related to rounding errors,
Generally $\varepsilon \approx \sigma_1 \varepsilon_{mach}$ ← 'machine precision'

- Cond. & Stab. ie 9/30, ② 1.1 pages
- Floating Pt. 9/30, ④ 0.7 pages.

break. ~~~.

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got to here. Lec 4

Stability (§14) : getting 'right' answer even if not exact

fix: problem $f : X \to Y$ eg. $y = \sin(x)$ or y is soln. to $Ay = b$ (here x 'data' is A, b).
floating pt. system.
algorithm for f, also map $X \to Y$.

data x
round to $fl(x)$
call $\tilde{f}(x)$ ↺ apply alg.

Relative error of computation $\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|}$ ← alg. good if this is $O(\varepsilon_{mach})$ 'on the order of', eg. $10^2 \varepsilon_{mach}$ ok. $10^8 \varepsilon_{mach}$ not.

skip [Formally: $O(\varepsilon_{mach})$ means $\le C\varepsilon_{mach}$ as $\varepsilon_{mach} \to 0$ for some C, uniformly over all data $x \in X$.]

Defns.

Alg <u>Stable</u> if $\forall x \in X$ $\frac{\|\tilde{f}(x) - f(\tilde{x})\|}{\|f(\tilde{x})\|} = O(\varepsilon_{mach})$ for some $\tilde{x}$ w/ $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{mach})$

nearly right ans. to nearly right qu.

Stronger: <u>Backward stable</u> : $\tilde{f}(x) = f(\tilde{x})$ for some $\tilde{x}$, $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{mach})$

<u>exactly</u> right ans to nearly right qu.

(15)
Eg. is $\ominus$ bkw stable?

problem is $f(x_1, x_2) = x_1 - x_2$
alg is $\tilde{f}(x_1, x_2) = fl(x_1) \ominus fl(x_2)$
$= [x_1(1+\varepsilon_1) - x_2(1+\varepsilon_2)](1+\varepsilon_3)$ ← for $|\varepsilon_1|, |\varepsilon_2|, |\varepsilon_3| \le \varepsilon_{mach}$
$= x_1(1+\varepsilon_4) - x_2(1+\varepsilon_5)$ for $\varepsilon_4, \varepsilon_5 \le 2\varepsilon_{mach} + O(\varepsilon_{mach}^2)$

ie exact for some data relatively close to x_1, x_2.

is $f(x) = x \ominus 1$ bkw st?

$\tilde{f}(x) = [x(1+\varepsilon_1) - 1](1+\varepsilon_2) = x(1+\varepsilon_3) - 1$ but $\varepsilon_3 = O(\varepsilon_{mach}) + \frac{1}{x} O(\varepsilon_{mach})$

so not bkw st. as $x \to 0$, but is stable.

some algs are crush

Take-home msg: algs in Matlab (Lapack, etc) for solving $Ay = b$ <u>are</u> bkw. stable:

$m = n$: QR is Thm 16.2

nonsing. sq.-sys. Gaussian elim w/ partial pivoting is, Ch. 22 (if avoid incredibly rare pathological matrices)

$m \geq n$:

least-squares soln, ie find x s.t. $\|Ax - b\|$ minimized. is bkw st. via SVD: (Thm 19.4)

$A = U \Sigma V^*$

$x = \underbrace{A^+}_{\text{pseudoinverse}} b = V^* \Sigma^{-1} U b$

[See NLA Ch. 6–23]

Note this means: The computed soln. $\tilde{y}$ to $Ay = b$ satisfies $(A + \delta A)\tilde{y} = b$ exactly, for some δA w/ $\frac{\|\delta A\|}{\|A\|} = O(\varepsilon_{mach})$

A plays role of data x.

y " answer f.

However, $\frac{\|\tilde{y} - y\|}{\|y\|}$ may <u>not</u> be small, ie y itself not accurate!

Such is backward stability: it's as good as could hope for! since A has roundoff.

How accurate is y? Need acc. of any bkw. stable alg:

$\star$ Thm (15.1): if cond # is κ for problem $f(x)$, alg. is bkw st, and computer obeys floating point axioms, then rel. errs. sat.

$$\frac{\|\tilde{f}(x) - f(x)\|}{\|f(x)\|} = O(\kappa(x)\, \varepsilon_{mach})$$

so, error norm is κ times worse than ε_{mach}. If $\kappa \geq 10^{+16}$ you loose <u>all</u> digits of y! But, still it holds that $(A + O(\varepsilon_{mach}))\tilde{y} = b$.!

pf: by defn. $\tilde{f}(x) = f(\tilde{x})$ (a) for some $\frac{\|\tilde{x} - x\|}{\|x\|} = O(\varepsilon_{mach})$ (b)

Defn of κ:

$$\frac{\|f(\tilde{x}) - f(x)\|}{\|f(x)\|} \leq \begin{pmatrix} \kappa \\ + o(1) \end{pmatrix} \frac{\|\tilde{x} - x\|}{\|x\|}$$

since not infinitesimal.

then sub. in (a) & (b)