Nyström beautifully efficient, & inherits the convergence of the quadr. scheme applied to $k(s,\cdot)u(\cdot)$, for certain well-behaved ops K, called 'compact.'

- If drop I in (LS) can apply to 1^{st}-kind (HW4), solving for $u_j^{(n)}$, but there is no interpolation formula (N).

Compact operators: may have ∞-dim range but behave 'like' finite-dim. ops.

consider funcs f in eg $C[a,b]$ as points in topological space, $X = C[a,b]$, w/ metric (norm) ('length') $\|f\|$, eg. $\|f\|_\infty = \sup_{s\in[a,b]} |f(s)|$

- seq $(f_n)_{n=1,2,\dots}$ bounded if $\|f_n\| \le C$ $(< \infty)$ $\forall n = 1,2,\dots$ note: seq. goes on forever (long time)
- seq. (f_n) converges to $f \in X$ if $\forall \varepsilon > 0$ no matter how small, $\exists N$ st. $\|f_n - f\| < \varepsilon$ $\forall n \ge N$

Say f_n were points in finite-dim space, eg $\mathbb{R}^m$, we'd apply Bolzano–Weierstrass.

Thm: in finite-dim normed lin. space, every bnded seq. contains a convergent subseq. ← also goes on forever

Eg. try it in $\mathbb{R}$: the only way to avoid having some limit pt is to escape to $\pm\infty$.

B-W proof uses this fact successively on each coord $1, 2 \dots m$ in $\mathbb{R}^m$.

However ∞-dim space such as $C[a,b]$, $L^2[a,b]$, etc. are fundamentally different.

Eg. seq $\{\sin(nx)\}_{n=1}^\infty$ is bounded in $C[0,2\pi]$ or $L^2[0,2\pi]$, but has no convergent subseq.

Eg. any other o.n.b.

E.g. ℓ_2: space of sequences $\{a_1, a_2, \dots\}$ with $\sum_{j=1}^\infty |a_j|^2 < \infty$ has o.n.b. $\{1,0,0,\dots\}$, $\{0,1,0,\dots\}$, $\{0,0,1,\dots\}$ etc.

Defn: Lin. op. $K: X \to Y$ between normed lin. spaces X & Y is compact if given any bounded seq (x_n) in X, the seq. (Kx_n) contains a convergent subseq. ← like in 'as if' finite-dim.

(Ie K maps bounded sets → 'precompact sets' (which must contain convergent subseq's).)

- Eg. if K has bounded finite-dim range, B-W $\Rightarrow$ (Kx_n) has conv. sub.seq, so K cpt.

eg. $(Ku)(s) = \int_a^b \underbrace{\sum_{i=1}^N \alpha_i(s)\beta_i(t)}_{\text{finite sum of rank-1 kernels}} u(t)\,dt$

- Conversely $K = Id$ with $X = Y$ ∞-dim spaces maps (bounded seq) $(x_n) \to (x_n)$ so need not contain conv. subseq, not cpt.

ie Id is compact iff Y finite-dim.

Thm: compact ops are bounded. pf suppose not, then $\exists$ seq (x_n) with $\|x_n\| = 1$ $\forall n = 1,2,\dots$ st. $\|Kx_n\| > n$. But this cannot contain convergent subseq. $\to K$ not compact.

But, bounded $\not\Rightarrow$ compact, viz. Id.

K compact = finite-dim, or ∞-dim in a way that components made arbitrarily small as dims $\to \infty$.

M116 . lec 9. still need final ½ hr on Ascoli thm. ① Thurs. 10/23/08.

· Coding tips. · print defn/fig & spectrum of cpt ops, eg. vs spec of $(K u)(s) = s u(s)$ which is cont.

· $\|K\|_2$ upper bnd via Hilbert–Schmidt (② 10/21), then $k(s,t) = \frac{1}{|s-t|^\gamma}$ is H-S for $0<\gamma<1/2$

· Compactness. (③ 10/21.)

Why compactness useful?

1) Thm (Fredholm Alternative). Let $K: X \to X$ be compact on normed lin. space X.
Then either i) for each $f \in X$, $(I-K)u = f$ has unique soln. $u \in X$
or ii) homog. eqn $(I-K)u = 0$ has nontrivial soln. ← the alternative.
If case i) holds, $(I-K)^{-1}$, whose existence is asserted there, is bounded.

· Ie, $(I-K)$ surjective (onto) iff injective (triv. nullspace). Note: eg. left-shift op. on l_2 has nontriv. nullsp, but also i) holds, in contrast.

Cf. invertible matrix thm: $A\vec{x} = \vec{b}$ soln exists, unique, iff $A\vec{x} = \vec{0}$ has only triv. soln.

· Means (I + compact) behaves like finite-dim square matrix (either invertible or not).

Useful: 1) if IE $(I-K)u = 0$ has only triv. soln, ie 1 is not an eigval. of K, then 2nd kind IE soln always exists, uniq.

2) we can prove convergence rate of Nyström method for compact K is $\|u^{(n)} - u\|_\infty \le C \underbrace{\|Ku - K_n u\|_\infty}$ quadrature error applied to $k(s,\cdot)u(\cdot)$

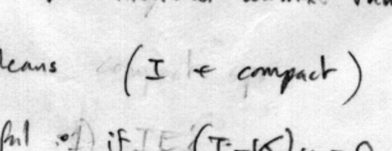
Defn: (x_n) 'Cauchy' convergent if $\forall \varepsilon > 0$, $\exists N$ st. $\|x_n - x_m\| \le \varepsilon$ for all $n,m > N$. "all pairs get arbitrarily close".
· $(x_n) \to x \Rightarrow$ Cauchy.
· if Cauchy seq. always convergent to an $x \in X$, X is complete (called Banach space), eg $C[a,b]$ in $\|\cdot\|_\infty$, $L^2[a,b]$, etc. not $C[a,b]$ in $\|\cdot\|_2$

Tests for compactness:

1) Thm: Let $K_1, K_2, \dots$ be seq. of compact ops $X \to Y$ (Y complete) st. $\lim_{n\to\infty} \|K_n - K\| = 0$, then K compact.

Rmk: eg. cpt if seq. of finite-dim ops which conv. in norm to it, eg. $u_j \to \frac{1}{j} u_j$ in l_2: K_n is $u_j \to \frac{1}{j} u_j$ $j \le n$, $u_j \to 0$ $j > n$ } so $K \leftarrow K_n$

Pf: let (x_n) bdded seq. $\exists$ subseq (x_{1n}) st. $(K_1 x_{1n})$ convergent.
$\exists$ subseq of this, (x_{2n}) st. $(K_2 x_{2n})$ conv. , .. etc.



choose 'diagonal subseq' x_{nn} : for each $k \in \mathbb{N}$, $(K_k x_{nn})$ conv. since subseq of x_{kn}.

If can show $(K x_{nn})$ conv. too, we've proved K cpt since for any (x_n) we found $K x_n$ has conv subseq.

Let $\varepsilon > 0$: $\exists k$ st. $\|K - K_k\| \le \frac{\varepsilon}{3M}$ where $M = \sup_n \|x_n\|$

Let k_1 be st. $\|K_k x_{nn} - K_k x_{mm}\| \le \varepsilon/3$ $\forall n,m > k_1$ (since $K_k x_{nn}$ conv., it's Cauchy)

then $\|K x_{nn} - K x_{mm}\| \le \underbrace{\|K x_{nn} - K_k x_{nn}\|}_{\varepsilon/3} + \underbrace{\|K_k x_{nn} - K_k x_{mm}\|}_{\varepsilon/3} + \underbrace{\|K_k x_{mm} - K x_{mm}\|}_{\varepsilon/3} \le \varepsilon$.

so $(K x_{nn})$ Cauchy $\Longrightarrow$ conv. to element in Y by completeness.

Ask who's looked at each others' work? Who's looked at debriefings? 10/23. (9)

— show matlab examples?

Coding tips:

(15 mins).

- if have param such as max n $(n=1\cdots 20)$ set it once at start of code & have all ~~diff~~ reference this.

- switches in code to avoid duplication (don't make many nearly-identical codes; more errors, harder to maintain, read). fncs. at start. use handles. $f=@(x)\ldots$ Eval. later but never need to type quadr. schemes.

- modular: $[x\ w] = \text{gauss}(n)$ — x nodes, w wei:, n order. defines a <u>uniform</u> interface. have other quadr. schemes return in same way: $[x\ w] = \text{comptrap}(n)$, equi(newtoncotes(~~~~ n)

main code eg Nyström: [loop over n … uses x, w.] $\xrightarrow{n}$ quadr. $\xrightarrow{(x,w)}$

$\boxed{\text{Lec. } 10}$

(Finish cpt) 10/28/08. (2)

Cpt ops K: a) Fredholm Alt: unique solvability of 2^{nd} kind $(I-K)u=f$ if $\text{Nul}(I-K)=\{0\}$... What means for spectrm of K? 1 is not eigenvalue. ($\lambda=1$: $Ku=u$ for some u).

b) Nyström conv. at same rate as quadr. scheme.

How to tell it cpt:

2) Fredholm K w/ $k\in C[a,b]^2$ is compact on $C[a,b]$.

Pf: i) show that K maps bounded seq (u) to <u>equicontinuous</u>, bdd seq.

eg. $u_n(t)=x^n$ — continuous on $[0,1]$ for each n, not equicontinuous on $[0,1]$ — why not?

$\subset$ (u_n) set of fncs: $\forall\varepsilon>0\ \exists\delta$ st. $\forall n$, $|u_n(x)-u_n(y)|<\varepsilon$ if $|x-y|<\delta$ key: uniformity over n.

ii) Then by amazing Arzela-Ascoli thm such sets are compact (seq. contain conv. subseq.) ... proof by diagonalization & $\varepsilon/3$ tricks.

Show i) only: $|(Ku)(s)| \le M|b-a| \max_{s,t}|k(s,t)|$ where $M=\sup_n\|u_n\|$.

k cont on square: $\forall\varepsilon>0\ \exists\delta$ st. $|k(s,t)-k(r,t)| \le \frac{\varepsilon}{M(b-a)}$ whenever $|s-r|<\delta$ where $M=\sup_n\|u_n\|$.

Then $|(Ku)(s)-(Ku)(r)| = \left|\int_a^b (k(s,t)-k(r,t))u_n(t)\,dt\right| \le |b-a| \max_{|s-r|<\delta}|k(s,t)-k(r,t)|\,\|u\|_\infty \le \varepsilon$

$\Rightarrow$ seq (Ku_n) equicontinuous.

Spectral thm for compact ops: K compact: has spectrum of discrete, finite-multiplicity eigenvalues, with 0 the only limit pt.

$\lambda_2 \cdots \lambda_4\ \lambda_3\ \lambda_1$ — no nasty continuous spectrum, etc.

since $\lambda_n\to 0$. think of map to ellipsoid w/ shrinking semiaxes. ball

($\lambda_n=\sigma_n$ SVD, if Hermitian self-adjoint)

Lec 10 ① 10/28/08.

· finish cpt ops.

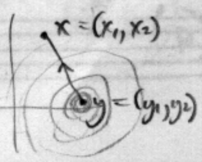
plan: finish cpt. (20 mins).
Fund soln. in 2d, Δ kills
Div. Thm & $\partial/\partial n$.
GT1, 2.
? prove GRF.
write as layer pots, define LPs, as int. op.
· formula for $\frac{\partial \Phi}{\partial n_y}(x,y)$.
· change-of-variables for $y(t)$ parametric curve.

Tools for Laplace's Eqn.

$$\underbrace{\Delta}_{\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}} u = 0 \iff u \text{ a harmonic func.}$$

steady-state heat flow, diffusion, electrostatics,

1) Fundamental soln. $\Phi(x,y) = -\frac{1}{2\pi}\ln|x-y|$ $\quad x,y \in \mathbb{R}^2$



aka 'free-space Greens func': will give integral kernel for solution op.

Fact: for each $y\in\mathbb{R}^2$, $\Phi(\cdot,y)$ is harmonic in $\mathbb{R}^2\setminus\{y\}$

$\mathbb{R}^2$ ie $\Delta_x \Phi(x,y) = 0 \quad \forall x\neq y$.

Check: set $y=0$ without loss of generality, then $\frac{\partial}{\partial x_1}\ln|x| = \frac{1}{2}\frac{\partial}{\partial x_1}\ln(x_1^2+x_2^2)$

$$= \frac{1}{2(x_1^2+x_2^2)}\cdot 2x_1 = \frac{x_1}{|x|^2} \quad x\neq 0 \quad \text{etc.}$$

· since ~~x↔~~y symm, $\Delta_y \Phi(x,y) = 0 \quad \forall y\neq x$.

2) Divergence Thm: $\Omega\subset\mathbb{R}^2$ open bounded domain w/ boundary $\partial\Omega$, $\vec{a} = \begin{pmatrix} a_1(x)\\ a_2(x)\end{pmatrix}$ vector field cont. on $\partial\Omega$ & C^1 cont inside,

$$\int_\Omega \underbrace{\vec{\nabla}\cdot\vec{a}}_{\operatorname{div}\vec{a} := \frac{\partial a_1}{\partial x_1}+\frac{\partial a_2}{\partial x_2}} \underbrace{dx}_{\text{area}} = \underbrace{\int_{\partial\Omega} \hat{n}\cdot\vec{a}\ \underbrace{ds}_{\text{arclength on } \partial\Omega}}_{\text{outward flux through } \partial\Omega}$$

$\Omega \to \hat{n}$ outward unit normal.

$\int \hat{n}(y)\cdot\vec{a}(y)\,ds_y$ explicitly.

· Ω may have corners.
· everything comes from these two.

Greens Thms: u,v be sufficiently smooth funcs defined in $\bar{\Omega}$ ← closure $\Omega+\partial\Omega$

$$\int_\Omega (u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v)\,dx = \int_{\partial\Omega} u \boxed{\frac{\partial v}{\partial n}}\,ds \qquad \text{(GT1)}$$

$=: v_n := \hat{n}\cdot\vec{\nabla}v$

$$\int_\Omega u\Delta v - v\Delta u \ dx = \int_{\partial\Omega} u v_n - v u_n \ ds \qquad \text{(GT2)}$$

pf: do as WS?
$\vec{\nabla}\cdot(u\vec{\nabla}v) \overset{\text{prod rule}}{=} u\Delta v + \vec{\nabla}u\cdot\vec{\nabla}v$.
apply Div Thm to $\vec{a} =$ to get GT1.
GT2 by subtracting GT1 with $u\leftrightarrow v$ from itself.

· sufficient smoothness is $\partial\Omega$ having C^1 smooth bdry, $u,v\in C^2(\bar{\Omega})$, however can be loosened to Ω w/ corners.

Corollary: setting $u\equiv 1$ in GT1 ; If v harmonic in Ω then $\int_{\partial\Omega} v_n\,ds = 0$ "zero-flux"

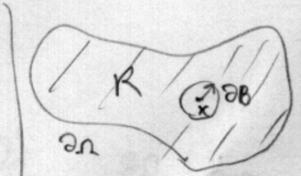
2

Green's Representation Formula: Let u be harmonic in Ω, then for each $x \in \Omega$,

$$u(x) = \int_{\partial\Omega} u_n(y)\,\Phi(x,y) - u(y)\frac{\partial\Phi}{\partial n_y}(x,y)\,ds_y$$

- ie interior values from integral kernels acting on bdry funcs. u, u_n.

Pf: $\partial B(x;r)$ = circle rad. r about x.



in $R := \Omega \setminus \underbrace{\overline{B}(x;r)}_{\text{closed ball}}$ $\Phi(x,y)$ harmonic as func. of y.

GT2 in R for u, $v = \Phi(x,\cdot)$:

$$\int_R u\,\underbrace{\Delta_y \Phi(x,y)}_{0} - \Phi(x,y)\,\underbrace{\Delta u}_{0}\,dx = \int_{\partial R} u(y)\frac{\partial \Phi(x,y)}{\partial n_y} - \Phi(x,y)\frac{\partial u}{\partial n}(y)\,ds_y$$

$\partial R = \partial\Omega + \partial B$. ← inwards-pointing $\hat{n}$.

so
$$\int_{\partial\Omega} \Phi(x,y)\,u_n(y) - \frac{\partial\Phi}{\partial n_y}(x,y)\,u(y)\,ds_y = \int_{\partial B(x;r)} \underbrace{\frac{\partial\Phi(x,y)}{\partial n_y}}_{\to\, +\frac{1}{2\pi r}} u(y) - \underbrace{\Phi(x,y)}_{\frac{-1}{2\pi}\ln r}\,u_n(y)\,ds_y$$

y on ∂B

$$= \frac{1}{2\pi r}\underbrace{\int_{\partial B} u(y)\,ds_y}_{=\,2\pi r\,\cdot\, u(y) \text{ for some } y\in\partial B \text{ by MVT.}} + \frac{1}{2\pi}\ln r \cancel{\int_{\partial B} u_n(y)\,ds_y}$$

by zero-flux

Finally take $\lim_{r\to 0}$, get $\lim_{r\to0} u(y)\big|_{y\in\partial B} = u(x)$.

Cor: since Φ & $\frac{\partial\Phi}{\partial n_y}$ analytic funcs of 1st variable, u analytic in Ω regardless how nasty (discont.) the bdry data u, u_n is.

Given 'density' func σ on $\partial\Omega$,
Define single-layer integral op $(S\sigma)(x) := \int_{\partial\Omega} \Phi(x,y)\,\sigma(y)\,ds_y$

double- $(D\tau)(x) = \int_{\partial\Omega} \frac{\partial\Phi(x,y)}{\partial n_y}\tau(y)\,ds_y$

then GRF says $u = S\sigma + D\tau$ if choose $\sigma = u_n|_{\partial\Omega}$
$\tau = -u|_{\partial\Omega}$

Other consequences of GRF: let u be harmonic,

Choose $\Omega = B(x;r)$ then $u(x) = \ln r \cancel{\int_{\partial\Omega} u_n(y)\,ds_y} - \frac{1}{2\pi r}\int u(y)\,ds_y$:= value at center = mean value of u over $\partial B(x;r)$

Zero flux

(2) Maximum principle).

③

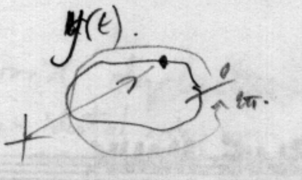
1) Computation :

$\mathbb{R}^2$.

given $x, y, \hat{n}_y$: (unit vec.)

$$\frac{\partial \Phi(x,y)}{\partial n_y} = -\frac{1}{2\pi}\, \hat{n} \cdot \nabla_y \ln|x-y| \qquad \frac{(x-y)}{|x-y|^2}$$

2) $\partial\Omega$ param by $y(t)$, $t \in [0, 2\pi)$:



for some bdry func g

$$\int_{\partial\Omega} g(y)\, ds_y = \int_0^{2\pi} g(y(t))\, \underbrace{|y'(t)|}_{\text{speed}}\, dt.$$