M116 lec. 3 (1) 3/27/14

Recall $\kappa(\vec{x}) = \dfrac{\|J(x)\|}{\|f\|/\|x\|}$ rel. cond. num.

Cond. of ~~basic~~ basic ops:

- $f(x) = 5x$ $(m=n=1)$ $J=5$ $\kappa = \dfrac{|J|}{5x/x} = \dfrac{5}{5} = 1$.
- $f(x) = x^{\alpha}$ $J = \alpha x^{\alpha-1}$ $\kappa = \dfrac{|\alpha x^{\alpha-1}|}{x^{\alpha}/x} = |\alpha|$. well cond for $|\alpha| \lesssim 10^3$
- $f(x_1,x_2) = x_2 - x_1$ subtraction $J = [-1 \;\; 1]$ so $\|J\|_2 = ?\ \sqrt{2}$

  $\kappa = \dfrac{\sqrt{2}\ \|x_1^2+x_2^2\|}{|x_2-x_1|} \to \infty$ if $x_2 \to x_1 \neq 0$.
  ill cond.
- $f(x) = \sin x$ $J(x) = \cos x$ $\kappa = \dfrac{|\cos x|}{|\sin x|/|x|} =$ for $x \to n\pi$, $n \in \mathbb{Z} \setminus \{0\}$.

  but even not nr. roots, if $x \sim 10^{100}$, v. ill cond.

- finding roots given poly coeffs: κ large.
- finding root near another for any smooth func:  $\kappa \sim \dfrac{1}{\delta}$. Large
- Eigvals of non-normal ($\Rightarrow$ nonsymm) matrices:

  eg $A = \begin{bmatrix} 1 & 10^3 \\ & 1 \end{bmatrix}$ vs $\begin{bmatrix} 1 & 10^3 \\ 10^{-3} & 1 \end{bmatrix}$ $\|\delta x\| = 10^{-3}$

  $\lambda \overset{?}{=} \{1,1\}$ $\lambda = \{0,2\}$ $\|\delta f\| \approx 1$

  But for normal (eg symm.) A, $\kappa = 1$ for eigvals. so $\hat{\kappa} \approx 10^3$, $\kappa \approx 10^6$.
- matvec $f(x) = \underbrace{A}_{m\times n}x$; $J = ?\ A$ itself

  $\Rightarrow \kappa = \dfrac{\|J\|}{\|Ax\|/\|x\|} = \|A\| \cdot \dfrac{\|x\|}{\|Ax\|}$ — how big can be? biggest is $\begin{pmatrix}\text{minimum growth factor} \\ \text{of } x \mapsto Ax\end{pmatrix}^{-1}$

If $m=n$ (square), $\dfrac{\|x\|}{\|Ax\|} \leq \|A^{-1}\|$ why? $\|x\| = \|A^{-1}Ax\| \leq \|A^{-1}\|\|Ax\|$
$\uparrow$ ∵ defn. 2-norm.

So, always $\kappa \leq \|A\|\|A^{-1}\| =: \text{cond}(A)$ ⇐ called "cond # of matrix A": A singular, $\kappa \to \infty$.

but if say x is in max growth factor direction, $\dfrac{\|x\|}{\|Ax\|} \approx \dfrac{1}{\|A\|}$ & $\kappa(x) = ?\ 1$ only.
When is ~~$\kappa(x)$~~ $\kappa(x)$ worst (largest)? x in min-growth direc.

- square system linear solve: What is input data for solution to $(Ax=b)$? b What's output? x So $f(b) = x = A^{-1}b$

if A, do later!

[PTO]

M116 lec 3. ① Tu - 4/1/14.

- papers : Larry = ? others, on board.
  - dates: ~~[illegible]~~ start Apr 18 Th.
- week 3: I'll present one. eg GMRES
  [feedback slips at end each lec: chance to respond].

finish lec 2:

recall. $f(x)$ is a problem.

abs. cond # $\hat\kappa(x) := \sup_{\delta x} \frac{\|\delta f\|}{\|\delta x\|} = \|J(x)\|$ ← ? matrix 2-norm

(jacobean of f)

--> note ③ 3/27/14

SVD: (singular value decomp.) formalizes above.

Work in $\mathbb{C}^m$: $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \end{pmatrix}$ $x^* = [\bar{x}_1, \bar{x}_2 \cdots]$ Hermitian / conjugate ~~transp~~.

(complex conj.)

2-norm. $\|x\| = \sqrt{x^* x}$

Def $q_1, \cdots q_m \in \mathbb{C}^n$ form o.n.b. if $q_i^* q_j = 0$ $i \neq j$, $\|q_i\| = 1$.

matrix $Q = [q_1 \cdots q_m] \in \mathbb{C}^{m\times m}$ unitary, $Q^*Q = ?$ I_m } $Q^{-1} = Q^*$.

$Q^* b$ = coeffs of expansion of b in the o.n.b. So $QQ^* = ?$ I_m by inverse: so rows also o.n.b.

HW: show $\|QA\| = \|A\|$.

- Rect. matrix w/ o.n.b. cols. ? $m \boxed{Q}_n$ $n > m$? not poss ⇒ $n < m$.

$Q^* Q = I_n$.

$QQ^* \in \mathbb{C}^{m\times m}$ = ? orthog. projector onto Ran Q.

why? if $b \in \text{Ran}\,Q$, $Q^* b$ = coeffs ⇒ $QQ^* b = b$

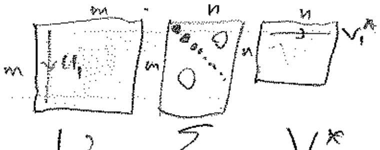
if $b \perp \text{Ran}\,Q$, $Q^* b = 0$. ⇒ $QQ^* b = 0$.

Thm (SVD: 4.1 in NLA): every $A \in \mathbb{C}^{m\times n}$ can be written $A = U\Sigma V^*$

where U, V unitary,

$\Sigma \in \mathbb{R}^{m\times n}$ diagonal w/ non-neg. entries

'sing. vals' $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_{\min(n,m)} \geq 0$.

= 

U rot into output o.n.b.! Σ stretch along coords V^* rot into coeffs of input o.n.b.

If $\{\sigma_j\}$ simple, Decomp. unique up to phases of cols of U, V.

As before max growth input direc is? v_1 Output direc.? u_1 Min growth factor ? last. $\sigma_{\min(n,m)}$

Output extremal direcs. u_j orthog (by geom), amazingly input v_j also orthog $v_2 \perp v_1$! (pf: NLA Ch. 4)

Cond (A) = ? σ_1/σ_n ($n\times n$) $r = \text{rank}(A)$ = ? $\#\{\sigma_j > 0\}$. If some σ's $= 0$, rank-deficient