Math 116 : Homework 3

due Tues Feb 21

Some of this HW is with pencil and paper (don't bother putting that online). Please label your plots. Qu. 1 is a series of refreshing little warm-ups; each part requires only a few lines of code (in Matlab, at least). Qu. 2 is the combined-field modification to scattering BIE (displaced Qu. 3 from HW2)—in some ways the culmination of the techniques from this course, so do do it, ask if stuck. Qu. 3 is simpler than BIE methods, and I think you'll enjoy it.

1. Trigonometric interpolation and integration of 1D periodic functions.

   (a) Show that the Lagrange polynomial for trigonometric interpolation $l_k(t) = \frac{1}{2n}\sum_m' e^{im(t-t_k)}$ can be written $\frac{1}{2n}\cot\left(\frac{t-t_k}{2}\right)\sin n(t-t_k)$.

   (b) Plot over the domain $[0, 2\pi]$ the function $l_5(t)$ when $n = 10$, and superimpose the grid points $\{t_k\}$ shown as blobs. Does it vanish in the right places? (Note: for plotting, you'll want to sample t at least 10 times finer than the grid points themselves).

   (c) With $n = 10$, compare the periodic analytic function $f(t) = e^{\sin x}$ with its trigonometric interpolation $f_n(t)$, by plotting the difference between the two. Beautiful, eh? [Hint: $f_n(t) = \sum_{j=1}^{2n-1} y_j l_j(t)$. Also note matlab arrays start at 1 but your points are labeled $j = 0 \cdots 2n-1$]. Repeat for the periodic continuous function $f(t) = |t-\pi|$. Comment.

   (d) Study the convergence with n of the uniform-grid, uniform-weight quadrature rule to approximate $I = \int_0^{2\pi}\left(1+\cos^2(t/2)\right)^{-1} dt$. How large does n need to be before 15 digits, which you should quote, are correct? Make a log-linear plot of difference from 'exact' vs $n = 1, 2, \ldots$, and add to your plot a line with slope given by the bound in the theorem of Lecture 6. [Hint: Remember there are $2n$ points when you apply the theorem]. How close is the convergence *rate* to this bound?

   (e) Here's an instructive example of what you need for Qu. 2. We wish to approximate $I = \int_0^{2\pi} K(t)\tau(t)dt$, with $K(t) = Y_0\left(3\sin\frac{|t-\pi|}{2}\right)$, and a given periodic analytic function τ. Let's split $K(t) = K_1(t)\ln\left(4\sin^2\frac{t-\pi}{2}\right) + K_2(t)$, so that both $K_1(t)$ and $K_2(t)$ are periodic analytic. This could be done many ways; do it so that $K_1(t) = bJ_0\left(3\sin\frac{|t-\pi|}{2}\right)$, where b is some constant. The use of the J_0 ensures analyticity. Use the small-argument asymptotics $Y_0(z) \sim \frac{2}{\pi}\left(\ln\frac{z}{2} + C\right) + O(z)$, and $J_0(z) \sim 1 + O(z^2)$ to find b. Also find the value $K_2(\pi) = \lim_{t\to\pi} K_2(t)$ (analogous to your 'diagonal' value of the kernel $M_2(s,s)$ from Lecture 9).

   (f) Choose $\tau(t) = e^{\sin t}$. Use uniform-weight quadrature to approximate $I_2 = \int_0^{2\pi} K_2(t)\tau(t)dt$. Note the log singularity is at π which is always the grid-point t_n, so at this point you'll need the value $K_2(\pi)$ you found above (at others use $K_2(t) = K(t) - bJ_0\left(3\sin\frac{|t-\pi|}{2}\right)\ln\left(4\sin^2\frac{t-\pi}{2}\right)$). Check $K_2(t)$ is indeed smooth, debug if not. Use logarithmic weights $R_j^{(n)}(\pi)$ from end of Lecture 10 to approximate $I_1 = \int_0^{2\pi} K_1(t)\tau(t)dt$. Check I_1 and I_2 converge exponentially by increasing n as earlier. What is $I = I_1 + I_2$, to 14 digits? [Hint: Euler constant C is `-psi(1)` in matlab. To debug your code I suggest comparing against a crude integration; your answer should be around 1.9]

2. 'Combined-field' modification to cure interior resonance problem for Helmholtz 2D scattering BIE.

(a) Modify your BIE code to fix the resonance problem from HW2 using a double-layer plus imaginary amount of single-layer representation, as described by Kress[1], see Eqn (1.12). This will require you splitting the logarithmic singularity according to Eqn (2.5), (2.6), and using the quadrature scheme given by Eqn (3.1). Essentially all you need to do is replace your $\tilde{K}$ matrix with the sum of two matrices as at the end of Lecture 9's notes (careful; there may be a factor 2π discrepancy in definition of $R_j^{(n)}$ between lectures 9 and 10). Check $M_2(s,t)$ and $M_1(s,t)$ are continuous. Choose $\eta = k$ as suggested.

(b) Redo the plot of condition number vs k around the resonant k range— has the problem gone away?

(c) Check that the new code gives the same exterior field $u(\mathbf{x})$ as the original one (by plotting the difference), for a nonsingular k value.

3. Finding Dirichlet eigenmodes with the Method of Patricular Solutions (MPS).

(a) Write a function which returns the j^{th} basis function ξ_j evaluated at either a point (x_1, x_2) or a list of such points. For basis functions use N plane waves: $N/2$ of the form $\sin(k\hat{d}\cdot x)$ then another $N/2$ of the form $\cos(k\hat{d}\cdot x)$, where the $N/2$ unit vectors $\hat{d}$ are spread uniformly in $[0,\pi)$.

(b) Make a function which fills the matrix $A_{jk} := \xi_k(y_j)$, $j = 1\cdots M$, $k = 1\cdots N$, where $\{y_j\}$ are the M boundary points of a mellower version of your trefoil shape you used for BIE: $f(\theta) = 1 + 0.1\cos(3\theta)$. Make a similar function which fills the matrix $A_{jk} := \xi_k(y_j^{(I)})$, $j = 1\cdots M$, $k = 1\cdots N$, where $\{y_j^{(I)}\}$ are M random interior points (they needn't even be uniformly sampled).

(c) Choose $N = 20$. Loop over $k \in [2, 10]$, computing at each k the generalized eigenvalue $\lambda(k)$ given by

$$F\mathbf{x} = \lambda(k)G\mathbf{x},$$

with $F = A^TA$ and $G = B^TB$. Plot λ vs k. At what k values does it vanish? [Bonus: what happens if N is varied?]

[1] Rainer Kress, "Boundary integral equations in time-harmonic acoustic scattering", Mathematical and Computer Modelling **15**(3–5) 229–243 (1991). This is available through Dartmouth's library system electronically, or through me.