Math 123 Homework Assignment #1
Due Friday, April 8th.

Part I:

1. Suppose that X is a normed vector space. Then X is a Banach space (that is, X is complete) if and only if every absolutely convergent series in X is convergent.

2. Let X be a normed vector space and suppose that S and T are bounded linear operators on X. Show that $\|ST\| \leq \|S\|\|T\|$.

3. Let X be a locally compact Hausdorff space. Show that $C_0(X)$ is a closed subalgebra of $C^b(X)$.

4. Let A be a unital Banach algebra. Show that $x \mapsto x^{-1}$ is continuous from $G(A)$ to $G(A)$. (Hint: $(a-h)^{-1} - a^{-1} = ((1 - a^{-1}h)^{-1} - 1)a^{-1}$.)

Part II:

5. Suppose that X is a compact Hausdorff space. If E is a closed subset of X, define $I(E)$ to be the ideal in $C(X)$ of functions which vanish on E.

(a) Let J be a closed ideal in $C(X)$ and let $E = \{\, x \in X : f(x) = 0 \text{ for all } f \in J \,\}$. Prove that if U is an open neighborhood of E in X, then there is a $f \in J$ such that $f(x) = 1$ for all x in the compact set $X \setminus U$.

(b) Conclude that $J = I(E)$ in part (a), and hence, conclude that *every closed* ideal in $C(X)$ has the form $I(E)$ for some closed subset E of X.

6. Suppose that X is a (non-compact) locally compact Hausdorff space. Let X^+ be the *one-point compactification* of X (also called the Alexandroff compactification: see [Kelly; Theorem 5.21] or [Folland, Proposition 4.36]). Recall that $X^+ = X \cup \{ \infty \}$ with $U \subseteq X^+$ open if and only if either U is an open subset of X or $X^+ \setminus U$ is a *compact* subset of X.

(a) Show that $f \in C(X)$ belongs to $C_0(X)$ if and only if the extension

$$\tilde{f}(\tilde{x}) = \begin{cases} f(\tilde{x}) & \text{if } \tilde{x} \in X, \text{ and} \\ 0 & \text{if } \tilde{x} = \infty. \end{cases}$$

is continuous on X^+.

(b) Conclude that $C_0(X)$ can be identified with the maximal ideal of $C(X^+)$ consisting of functions which 'vanish at ∞.'

7. Use the above to establish the following ideal theorem for $C_0(X)$.

Theorem: Suppose that X is a locally compact Hausdorff space. Then every closed ideal J in $C_0(X)$ is of the form

$$J = \{ f \in C_0(X) : f(x) = 0 \text{ for all } x \in E \}$$

for some closed subset E of X.

Part III:

8. Assume you remember enough measure theory to show that if $f, g \in L^1\big([0,1]\big)$, then

$$f * g(t) = \int_0^t f(t-s)g(s)\,ds \tag{3}$$

exists for almost all $t \in [0,1]$, and defines an element of $L^1\big([0,1]\big)$. Let A be the algebra consisting of the Banach space $L^1\big([0,1]\big)$ with multiplication defined by (3).

(a) Conclude that A is a commutative Banach algebra: that is, show that $f * g = g * f$, and that $\|f * g\|_1 \le \|f\|_1 \, \|g\|_1$.

(b) Let f_0 be the constant function $f_0(t) = 1$ for all $t \in [0,1]$. Show that

$$f_0^n(t) := f_0 * \cdots * f_0(t) = t^{n-1}/(n-1)!, \tag{4}$$

and hence,

$$\|f_0^n\|_1 = \frac{1}{n!}. \tag{5}$$

(c) Show that (4) implies that f_0 generates A as a Banach algebra: that is, $\mathrm{alg}(f)$ is norm dense. Conclude from (5) that the spectral radius $\rho(f)$ is zero for all $f \in A$.

(d) Conclude that A *has no nonzero complex homomorphisms.*

9. Here we want to give an example of a unital commutative Banach algebra A where the Gelfand transform induces and injective isometric map of A onto a proper subalgebra of $C(\Delta)$. For A, we want to take the *disk algebra*. There are a couple of ways that the disk algebra arises in the standard texts, but the most convenient for us is to proceed as follows. Let $D = \{\, z \in \mathbb{C} : |z| < 1 \,\}$ be the open unit disk. We'll naturally write $\overline{D}$ for its closure $\{\, z \in \mathbb{C} : |z| \leq 1 \,\}$, and $\mathbb{T}$ for its boundary. Then A will be the subalgebra of $C(\overline{D})$ consisting of functions which are holomorphic on D. Using Morera's Theorem, it is not hard to see that A is closed in $C(\overline{D})$, and therefore a unital commutative Banach algebra.[5] Notice that for each $z \in \overline{D}$, we obtain $\varphi_z \in \Delta$ by $\varphi_z(f) := f(z)$. We'll get the example we want by showing that $z \mapsto \varphi_z$ is a homeomorphism Ψ of $\overline{D}$ onto Δ. For convenience, let $p_n \in A$ be given by $p_n(z) = z^n$ for $n = 0, 1, 2, \ldots$, and let $\mathcal{P}$ be the subalgebra of polynomials spanned by the p_n.

(a) First observe that Ψ is injective. (Consider p_1.)

(b) If $f \in A$ and $0 < r < 1$, then let $f_r(z) := f(rz)$. Show that $f_r \to f$ in A as $r \to 1$.

(c) Conclude that $\mathcal{P}$ is dense in A. (Hint: show that $f_r \in \overline{\mathcal{P}}$ for all $0 < r < 1$.)

(d) Now show that Ψ is surjective. (Hint: suppose that $h \in \Delta$. Then show that $h = \varphi_z$ where $z = h(p_1)$.)

(e) Show that Ψ is a homeomorphism. (Hint: Ψ is clearly continuous and both $\overline{D}$ and Δ are compact and Hausdorff.)

[5] The maximum modulus principal implies that the map sending $f \in C(\overline{D})$ to its restriction to $\mathbb{T}$ is an isometric isomorphism of A onto a closed subalgebra $A(D)$ in $C(\mathbb{T})$. Of course, our analysis applies equally well to $A(D)$.

(f) Observe that if we use the above to identify Δ and $\overline{D}$, then the Gelfand transform is the identify on A, and A is a proper subalgebra of $C(\overline{D})$.

10. In this problem, we want to prove an old result to due Wiener about functions with absolutely converent Fourier series using the machinery of Gelfand theory. Recall that if $\varphi \in C(\mathbb{T})$, then the *Fourier coefficients* of φ are given by[6]

$$\check{\varphi}(n) := \frac{1}{2\pi} \int_{-\pi}^{\pi} \varphi(e^{it}) e^{-int}\, dt.$$

In some cases — for example if φ has two continuous derivatives — the Fourier coefficients are *absolutely convergent* in the sense that $n \mapsto \check{\varphi}(n)$ defines an element of $\ell^1(\mathbb{Z})$.[7]

We aim to prove the following:

Theorem: (Wiener) *Suppose the $\varphi \in C(\mathbb{T})$ never vanishes and has an absolutely convergent Fourier series. Then $\psi := 1/\varphi$ also has an absolutely convergent Fourier series.*

I suggest the following strategy.

(a) If f and g are in $\ell^1(\mathbb{Z})$, then their *convolution*, $f * g$ is defined by

$$f * g(n) = \sum_{m=-\infty}^{\infty} f(m) g(n-m).$$

Show that $f * g \in \ell^1(\mathbb{Z})$ (so that in particular, $f * g(n)$ is defined for all n), and that convolution makes $\ell^1(\mathbb{Z})$ into a unital, commutative Banach algebra. (Here, $1_{\ell^1(\mathbb{Z})} = \mathbb{1}_{\{0\}}$.[8])

(b) Let $\Delta = \Delta(\ell^1(\mathbb{Z}))$ be the maximal ideal space of $\ell^1(\mathbb{Z})$ equipped with its compact, Hausdorff Gelfand topology. If $z \in \mathbb{T}$, then define $h_z : A \to \mathbb{C}$ by

$$h_z(f) = \sum_{n=-\infty}^{\infty} f(n) z^n.$$

Show that $h_z \in \Delta$.

[6] I've used $\check{\varphi}$ in place of the traditional $\hat{\varphi}$ to distinguish it from the (other) Gelfand transform to be used in the problem.

[7] Recall that $\ell^1(\mathbb{Z}) = L^1(\mathbb{Z}, \nu)$, where ν is counting measure, is the set of functions $f : \mathbb{Z} \to \mathbb{C}$ such that $\lim_{N\to\infty} \sum_{n=-N}^{n=N} |f(n)| < \infty$.

[8] If S is a subset of X, I use $\mathbb{1}_S$ for the characteristic function of S, which takes the value 1 on S, and 0 otherwise.

(c) Let $w = \mathbb{1}_1 \in \ell^1(\mathbb{Z})$. If $h \in \Delta$, then show that $h = h_z$ where $z = h(w)$. (Hint: If $f \in \ell^1(\mathbb{Z})$, then $f = \sum_{n=-\infty}^{\infty} f(n)w^n$ in norm in $\ell^1(\mathbb{Z})$, where for example, $w^2 = w * w = \mathbb{1}_2$ and $w^{-1} = \mathbb{1}_{-1}$.)

(d) Show that $z \mapsto h_z$ is a homeomorphism Φ of $\mathbb{T}$ onto Δ. (Hint: Since both $\mathbb{T}$ and Δ are compact Hausdorff sets, it suffices to see that Φ is a continuous bijection. To show that Φ is continuous, observe that functions of the form $\sum_{n=-N}^{n=N} f(n)w^n$ are dense in $\ell^1(\mathbb{Z})$.)

(e) Since we can identify $\mathbb{T}$ with Δ, if $f \in \ell^1(\mathbb{Z})$, we will view the Gelfand transform of f as a continuous function on $\mathbb{T}$. (So that we write $\hat{f}(z)$ in place of $\hat{f}(h_z)$.) Show that if $\varphi = \hat{f}$ for some $f \in A$, then $\check{\varphi} = f$.

(f) Conclude that the image $\mathfrak{A}$ of $\ell^1(\mathbb{Z})$ in $C(\mathbb{T})$ under the Gelfand transform is exactly the set of φ in $C(\mathbb{T})$ whose Fourier coefficients are absolutely convergent. (That is, $\mathfrak{A}$ is the collection of $\varphi \in C(\mathbb{T})$ such that $n \mapsto \check{\varphi}(n)$ is in $\ell^1(\mathbb{Z})$.)

(g) Now prove Wiener's Theorem as stated above. (Hint: More or less by assumption, $\varphi = \hat{f}$ for some f in $\ell^1(\mathbb{Z})$. Show that f must be invertible in $\ell^1(\mathbb{Z})$ and consider the Gelfand transform of the inverse of f.)