Math 123 Homework Assignment #2
Due Friday, April 22

Part I:

1. Suppose that A is a C^*-algebra.

   (a) Suppose that $e \in A$ satisfies $xe = x$ for all $x \in A$. Show that $e = e^*$ and that $\|e\| = 1$. Conclude that e is a unit for A.

   (b) Show that for any $x \in A$, $\|x\| = \sup_{\|y\|\leq 1} \|xy\|$. (Do *not* assume that A has an approximate identity.)

2. Suppose that A is a Banach algebra with an involution $x \mapsto x^*$ that satisfies $\|x\|^2 \leq \|x^*x\|$. Then show that A is a Banach $*$-algebra (i.e., $\|x^*\| = \|x\|$). In fact, show that A is a C^*-algebra.

3. Let I be a set and suppose that for each $i \in I$, A_i is a C^*-algebra. Let $\bigoplus_{i\in I} A_i$ be the subset of the direct product $\prod_{i\in I} A_i$ consisting of those $a \in \prod_{i\in I} A_i$ such that $\|a\| := \sup_{i\in I} \|a_i\| < \infty$. Show that $\big(\bigoplus_{i\in I} A_i, \|\cdot\|\big)$ is a C^*-algebra with respect to the usual pointwise operations:

$$(a + \lambda b)(i) := a(i) + \lambda b(i)$$
$$(ab)(i) := a(i)b(i)$$
$$a^*(i) := a(i)^*.$$

We call $\bigoplus_{i\in I} A_i$ the *direct sum* of the $\{\, A_i \,\}_{i\in I}$.

4. Let A^1 be the vector space direct sum $A \oplus \mathbf{C}$ with the $*$-algebra structure given by

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu)$$
$$(a, \lambda)^* := (a^*, \bar{\lambda}).$$

Show that there is a norm on A^1 making it into a C^*-algebra such that the natural embedding of A into A^1 is isometric. (Hint: If $1 \in A$, then show that $(a, \lambda) \mapsto (a + \lambda 1_A, \lambda)$ is a $*$-isomorphism of A^1 onto the C^*-algebra direct sum of A and $\mathbf{C}$. If $1 \notin A$, then for each $a \in A$, let L_a be the linear operator on A defined by left-multiplication by a: $L_a(x) = ax$. Then show that the collection B of operators on A of the form $\lambda I + L_a$ is a C^*-algebra with respect to the operator norm, and that $a \mapsto L_a$ is an isometric $*$-isomorphism.)

5. In this question, ideal always means ‘closed two-sided ideal.’

(a) Suppose that I and J are ideals in a C^*-algebra A. Show that IJ — defined to be the closed linear span of products from I and J — equals $I \bigcap J$.

(b) Suppose that J is an ideal in a C^*-algebra A, and that I is an ideal in J. Show that I is an ideal in A.

6. Suppose that a and b are elements in a C^*-algebra A and that $0 \leq a \leq b$. Show that $\|a\| \leq \|b\|$. What happens if we drop the assumption that $0 \leq a$? (Hint: use Lemma Z.)

Part II:

7. Suppose that A is a unital C^*-algebra and that $f : \mathbf{R} \to \mathbf{C}$ is continuous. Show that the map $x \mapsto f(x)$ is a continuous map from $A_{\text{s.a.}} = \{ x \in A : x = x^* \}$ to A.

8. Prove Corollary AC: Show that every separable C^*-algebra contains a sequence which is an approximate identity. (Recall that we showed in the proof of Theorem AB that if $x \in A_{\text{s.a.}}$, and if $x \in \{ x_1, \dots, x_n \} = \lambda$, then $\|x - xe_\lambda\|^2 < 1/4n$.)

9. Suppose that $\pi : A \to B(\mathcal{H})$ is a representation. Prove that the following are equivalent.

(a) π has no non-trivial closed invariant subspaces; that is, π is irreducible.

(b) The commutant $\pi(A)' := \{ T \in B(\mathcal{H}) : T\pi(a) = \pi(a)T \text{ for all } a \in A \}$ consists solely of scalar multiples of the identity; that is $\pi(A)' = \mathbf{C}I$.

(c) No non-trivial projection in $B(\mathcal{H})$ commutes with every operator in $\pi(A)$.

(d) Every vector in $\mathcal{H}$ is cyclic for π.

(Suggestions. Observe that $\pi(A)'$ is a C^*-algebra. If $A \in \pi(A)'_{\text{s.a.}}$ and $A \neq \alpha I$ for some $\alpha \in \mathbf{C}$, then use the Spectral Theorem to produce nonzero operators $B_1, B_2 \in \pi(A)'$ with $B_1B_2 = B_2B_1 = 0$. Observe that the closure of the range of B_1 is a non-trivial invariant subspace for π.)

Part III:

10. As in footnote 1 of problem #8 on the first assignment, use the maximum modulus theorem to view the disk algebra, $A(D)$, as a Banach subalgebra of $C(\mathbf{T})$.[1] Let $f \in A(D)$ be the identity function: $f(z) = z$ for all $z \in \mathbf{T}$. Show that $\sigma_{C(\mathbf{T})}(f) = \mathbf{T}$, while $\sigma_{A(D)}(f) = \overline{D}$. This shows that, unlike the case of C^*-algebras where we have "spectral permanence," we can have $\sigma_A(b)$ a proper subset of $\sigma_B(b)$ when B is a unital subalgebra of A.

11. Suppose that U is an bounded operator on a complex Hilbert space $\mathcal{H}$. Show that the following are equivalent.

(a) U is isometric on $\ker(U)^\perp$.

(b) $UU^*U = U$.

(c) UU^* is a projection[2].

(d) U^*U is a projection.

An operator in $B(\mathcal{H})$ satisfying (a), and hence (a)–(d), is called a *partial isometry* on $\mathcal{H}$. The reason for this terminology ought to be clear from part (a).

Conclude that if U is a partial isometry, then UU^* is the projection on the (necessarily closed) range of U, that U^*U is the projection on the $\ker(U)^\perp$, and that U^* is also a partial isometry.

(Hint: Replacing U by U^*, we see that (b)$\Longleftrightarrow$(c) implies (b)$\Longleftrightarrow$(c)$\Longleftrightarrow$(d). Then use (b)–(d) to prove (a). To prove (c)$\Longrightarrow$(b), consider $(UU^*U - U)(UU^*U - U)^*$.)

[1] Although it is not relevant to the problem, we can put an involution on $C(\mathbf{T})$, $f^*(z) = \overline{f(\bar{z})}$, making $A(D)$ a Banach $*$-subalgebra of $C(T)$. You can then check that neither $C(\mathbf{T})$ nor $A(D)$ is a C^*-algebra with respect to this involution.

[2] A a bounded operator P on a complex Hilbert space $\mathcal{H}$ is called a *projection* if $P = P^* = P^2$. The term *orthogonal projection* or *self-adjoint projection* is, perhaps, more accurate. Note that $\mathcal{M} = P(\mathcal{H})$ is a *closed* subspace of $\mathcal{H}$ and that P is the usual projection with respect to the direct sum decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. However, since we are only interested in these sorts of projections, we will settle for the undecorated term "projection."