MATH 123 WINTER 2016
AUTOMORPHIC FORMS, REPRESENTATIONS AND C*-ALGEBRAS

DIARY

References

[**B**] *Automorphic forms and representations*, by D. Bump.
[**K**] *Introduction to the representation theory of groups*, by E. Kowalski.
[**tfb²**] *Crossed products of C*-algebras*, by D. P. Williams.
[**R**] *Functional analysis*, by W. Rudin.

Other sources were used, including **P. Garett's vignettes** and **W. Casselman's essays**.

Syllabus

I. Overview of representation theory, automorphic forms and applications
- *3 lectures + 1 guest lecture*

1. Group representations and harmonic analysis on homogeneous spaces
2. Selberg's $\frac{1}{4}$ Conjecture
3. L-functions and applications

II. Waveforms for cocompact lattices of $\mathrm{SL}(2,\mathbb{R})$ - *14 lectures*

4. Maass forms and the spectral problem, unbounded operators
5. Differential operators and Lie algebras
6. The Cartan decomposition and K-bi-invariant functions
7. Discreteness of the spectrum

III. Admissible and unitary representations of $\mathrm{SL}(2,\mathbb{R})$ - *7 lectures*

8. Admissible $(\mathfrak{g}, K)$-modules
9. Irreducible $(\mathfrak{g}, K)$-modules for $\mathrm{SL}(2,\mathbb{R})$
10. Unitarizability of admissible representations
12. The unitary dual of $\mathrm{SL}(2,\mathbb{R})$ and the solution of the spectral problem

IV. The noncommutative geometry point of view - *3 lectures*

11. Induced representations and Frobenius reciprocity for finite groups
13. Parabolic induction in the C*-algebraic framework and applications

Updated: March 7, 2016.

Lecture 1. Topological groups: Haar measure(s), modular function.
Group representations: examples, unitary representations, continuous representations. The left regular representation λ_G is unitary and continuous. Irreducible representations. Case of the torus: $\widehat{\mathbb{T}} = \{\chi_n \ , \ n \in \mathbb{Z}\}$ with χ_n acting on $\mathbb{C}_n = \mathbb{C}$ for every n. The L^2-theory of Fourier series gives a decomposition of the regular representation into irreducibles:

$$L^2(\mathbb{T}) \simeq \sum_{n\in\mathbb{Z}}^{\oplus} \mathbb{C}_n.$$

References: [**tfb²**, §1.3] and [**K**, §3.3-3.4].

Lecture 2. Fourier transform: $L^2(\mathbb{R}) \simeq \int_{\xi\in\mathbb{R}}^{\oplus} \mathbb{C}_\xi \, d\xi$ as a unitary representation of $\mathbb{R}$. General case of locally compact abelian groups: unirreps are one-dimensional, form a locally compact group (the Pontrjagyn dual $\widehat{G}$) and Fourier theory yields:

$$L^2(G) \simeq \int_{\chi\in\widehat{G}}^{\oplus} \mathbb{C}_\chi \, d\chi.$$

Definition of unitary equivalence, the unitary dual of a locally compact group:

$$\widehat{G}_u = \{\text{irreducible continuous unitary representations of } G\} / \text{unitary equivalence}.$$

Example: $\widehat{\mathrm{SO}(3)} = \{\mathcal{H}_\ell \ , \ \ell \in \mathbb{N}\}$ where $\dim \mathcal{H}_\ell = 2\ell + 1$. Summary of Peter-Weyl theory: for G compact, the regular representation decomposes as

$$L^2(G) \simeq \sum_{\pi\in\widehat{G}}^{\oplus} \dim(\mathcal{H}_\pi)\mathcal{H}_\pi$$

and the inversion formula reads

$$f = \sum_{\pi\in\widehat{G}} \dim(\mathcal{H}_\pi) \left(\mathrm{Tr}\, \pi * f\right).$$

General case of real reductive groups: Harish-Chandra proved the existence of (and determined) a measure μ on $\widehat{G}$ such that, for $f \in C_c(G)$,

$$f = \int_{\pi\in\widehat{G}} (\Theta_\pi * f) \; d\mu(\pi)$$

where each Θ_π is a distribution on G, generalizing the trace.
Case of $\mathrm{SL}(2,\mathbb{R})$(Bargmann, 1947): the unitary dual consists of

- the discrete series;
- the principal series and the limits of discrete series;
- the complementary series and the trivial representation.

[PICTURE]

The concrete Plancherel formula for $f \in C_c(\mathrm{SL}(2,\mathbb{R}))$ is

$$f = \sum_{n\in\mathbb{Z}} |n| \left(\Theta_n * f\right) + \frac{1}{4}\int_{-\infty}^{+\infty} \left(\Theta_{\nu_1} * f\right) \nu_1 \tanh\left(\frac{\pi\nu_1}{2}\right) d\nu_1 + \frac{1}{4}\int_{-\infty}^{+\infty} \left(\Theta_{\nu_2} * f\right) \nu_2 \coth\left(\frac{\pi\nu_2}{2}\right) d\nu_2.$$

Some representations (the complementary series) do not appear in the Plancherel formula. The ones that do are called *tempered* and form a closed subspace $\widehat{G}_r \subset \widehat{G}$.
The discrete series behave like representations of compact groups: the factor $|n|$ should be interpreted as a *formal dimension* for these representations, which are characterised by the fact that they are actual subrepresentations of the regular and that their matrix coefficients are square-integrable. The other tempered representations are only weakly contained in the regular and have almost square-integrable matrix coefficients.

Lecture 3. Quasi-regular representations: if G acts on a space X that carries a G-invariant Borel measure μ, then $L^2(X, \mu)$ is a unitary representation of G. Criterion for the existence of a G-invariant measure on a homogeneous space G/H: $\Delta_G|_H = \Delta_H$. In particular, this is satisfied when G is reductive and H is discrete.
Case of $G = \mathrm{SL}(2, \mathbb{R})$ and $H = \Gamma(N)$ or a congruence subgroup.

Theorem (Gelfand, Graev, Piatetski-Shapiro)**.** *As a unitary representation of G,*

$$L^2(\Gamma\backslash G) \simeq \mathcal{H}_1 \oplus \mathcal{H}_2$$

where

- *$\mathcal{H}_1$ is a direct sum indexed by a countable subset of $\widehat{G}$:*

$$\mathcal{H}_1 = \sum_{\pi}{}^{\oplus} m_\pi \mathcal{H}_\pi$$

- *$\mathcal{H}_2$ is a direct integral of principal series representations of G:*

$$\mathcal{H}_2 = \int_{\nu\in\mathbb{R}}^{\oplus} m_\Gamma \mathcal{H}_\nu \, d\nu$$

  where m_Γ only depends on Γ. In fact, $m_\Gamma = 0$ is Γ is a cocompact lattice. In general, it is equal to the number of cusps of Γ.

Selberg conjectured in 1965 that no complementary series occurs in $\mathcal{H}_1$ if $\Gamma = \Gamma(N)$.
K-fixed vectors in spherical representations are smooth and eigenfunctions of the hyperbolic Laplace operator. Conversely, to an automorphic form f with eigenvalue λ, one can associate a representation π_f and π_f is in the complementary series if and only if $\lambda < \frac{1}{4}$.
Selberg's $\frac{1}{4}$ Conjecture is still open in general but Selberg proved that $\mathrm{Sp}\,\Delta \subset \left[\frac{3}{16}, +\infty\right)$.

Reference: [**K**, §7.4].

Week 2

Lecture 4. Reciprocal sums of primes numbers, Dirichlet's Arithmetic Progression.
Euler product for Riemann's ζ function, estimates near 1. Dirichlet characters and associated L-series:

$$\sum_{n\geq 1} \frac{\chi(n)}{n^s}.$$

Proof of a special case of Dirichlet's Theorem: $\displaystyle\sum_{p\equiv 1[4]} \frac{1}{p}$ diverges.

Maass cusp forms. Periodicity and Fourier expansion:

$$f(z) = \sum_{n\in\mathbb{Z}} a_n(y) e^{2i\pi n x}$$

with $a_n(y) = c_n\sqrt{y}K_\nu(2\pi n y)$ where K_ν is a Bessel function. The corresponding L-function

$$L(s,f) = \sum_{n\geq 1} \frac{c_n}{n^s}$$

satisfies a functional equation.

References: [**B**, §1.9].

Lecture 5. Given a weight $k \in \mathbb{Z}$, the *Maass operators* on the Poincaré plane $\mathscr{H}$ are

$$R_k = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{k}{2} = (z-\bar{z})\frac{\partial}{\partial z} + \frac{k}{2}$$

and

$$L_k = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{k}{2} = -(z-\bar{z})\frac{\partial}{\partial \bar{z}} - \frac{k}{2}.$$

The *weight k non-Euclidean Laplacian* is

$$\Delta_k = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iky\frac{\partial}{\partial x}.$$

They are related *via*

$$-L_{k+2}R_k - \frac{k}{2}\left(1+\frac{k}{2}\right) = \Delta_k = -R_{k-2}L_k - \frac{k}{2}\left(1-\frac{k}{2}\right).$$

For each $k \in \mathbb{Z}$, the group $G = \mathrm{GL}(2,\mathbb{R})^+$ acts on the right on $C^\infty(\mathscr{H})$ by

$$f|_k g = \left(\frac{c\bar{z}+d}{|cz+d|}\right)^k f\left(\frac{az+b}{cz+d}\right)$$

where $g = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The Maass operators satisfy the following equivariance relations:

$$(R_k f)|_{k+2} g = R_k (f|_k g)$$
$$(L_k f)|_{k-2} g = L_k (f|_k g)$$
$$(\Delta_k f)|_k g = \Delta_k (f|_k g).$$

We will study the operators Δ_k in the context of Hilbert spaces, that is as *unbounded operators*. Adjoint of a densely defined operator.

References: [**R**, Chap. 13].

Lecture 6. Elementary properties of unbounded operators: domain of a sum, composition, associativity. Distributivity might fail: $T(R+S) \supset TR+TS$ in general.
A densely defined operator T is *symmetric* if $T \subset T^*$, that is

$$\langle Tx, y\rangle = \langle x, Ty\rangle$$

for all $x, y \in \mathcal{D}(T)$. It is *self-adjoint* if $T = T^*$.
Symmetric operators may or may not have self-adjoint extensions: example of $i\dfrac{d}{dx}$ on $L^2([0,1])$, with various domains, after Rudin [**R**, Chap. 13], [**B**, §2.1].
The measure $\dfrac{dx \wedge dy}{y^2}$ is $\mathrm{SL}(2,\mathbb{R})$-invariant (use Bruhat decomposition to shorten the verification). Green's formula for the Euclidean Laplacian $\Delta^e = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$:

$$\int_\Omega (g\Delta^e f - f\Delta^e g)\, dx \wedge dy = \int_{\partial\Omega} g\left(\frac{\partial f}{\partial x}dy - \frac{\partial f}{\partial y}dx\right) - f\left(\frac{\partial g}{\partial x}dy - \frac{\partial g}{\partial y}dx\right).$$

References: [**R**, Chap. 13], [**B**, §2.1].

Week 3

Lecture 7. The hyperbolic Laplacian $(\Delta_k, C_c^\infty(\mathscr{H}))$ is a symmetric operator on $L^2(\mathscr{H})$.
Let Γ be a subgroup of $\mathrm{SL}(2,\mathbb{R})$ acting discontinuously on $\mathscr{H}$, $\chi \in \mathrm{Hom}(\Gamma, \mathbb{T})$ a character, $k \in \mathbb{Z}$ a weight and define $C^\infty(\Gamma\backslash\mathscr{H}, \chi, k)$ as

$$\left\{ f \in C^\infty(\mathscr{H}) \quad , \quad \forall \gamma = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \Gamma \quad , \quad f(\gamma\cdot z) = \chi(\gamma)\left(\frac{c\bar{z}+d}{|cz+d|}\right)^{-k} f(z) \right\}$$

with the compatibility assumption $\chi(-I_2) = (-1)^k$.
If $f, g \in C^\infty(\Gamma\backslash\mathscr{H}, \chi, k)$, then $f\bar{g}$ is Γ-invariant and one can define

$$\langle f, g\rangle = \int_{\Gamma\backslash\mathscr{H}} f(z)\overline{g(z)}\,\frac{dx\,dy}{y^2}$$

and complete $C^\infty(\Gamma\backslash\mathscr{H}, \chi, k)$ into a Hilbert space, denoted by $L^2(\Gamma\backslash\mathscr{H}, \chi, k)$.
Behaviour of the Maass operators:

$$R_k : C^\infty(\Gamma\backslash\mathscr{H}, \chi, k) \longrightarrow C^\infty(\Gamma\backslash\mathscr{H}, \chi, k+2)$$

$$L_k : C^\infty(\Gamma\backslash\mathscr{H}, \chi, k) \longrightarrow C^\infty(\Gamma\backslash\mathscr{H}, \chi, k-2)$$

$$\Delta_k : C^\infty(\Gamma\backslash\mathscr{H}, \chi, k) \longrightarrow C^\infty(\Gamma\backslash\mathscr{H}, \chi, k)$$

and

$$\langle R_k\, f, g\rangle = \langle f, -L_k\, g\rangle$$

for f and g in spaces with appropriate weights.
It follows that Δ_k is a symmetric operator on $L^2(\Gamma\backslash\mathscr{H}, \chi, k)$.

Spectral problem (v.1): determine the spectrum of Δ_k on $L^2(\Gamma\backslash\mathscr{H}, \chi, k)$.

References: [**B**, §2.1].

Lecture 8. Definition of *Maass forms of weight* k as elements of $C^\infty(\Gamma\backslash\mathscr{H},\chi,k)\cap\mathrm{Sp}(\Delta_k)$. Generalities on Iwasawa decomposition and decompositions of Haar measure.
In the case of $G=\mathrm{SL}(2,\mathbb{R})$, every element g can be written uniquely as

$$g=\begin{bmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{bmatrix}\underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}_{R_\theta} \tag{1}$$

with $\theta\in\mathbb{R}/2\pi\mathbb{Z}$, $x\in\mathbb{R}$ and $y\in\mathbb{R}_+^\times$ and the Haar measure decomposes accordingly in these coordinates:

$$dg=\frac{dx\,dy}{y^2}\,d\theta.$$

Given a character $\chi\in\mathrm{Hom}(\Gamma,\mathbb{T})$, consider

$$L^2(\Gamma\backslash G,\chi)=\left\{f\in L^2(G)\quad,\quad\forall\gamma\in\Gamma\ ,\ f(\gamma\cdot z)=\chi(\gamma)f(z)\right\}.$$

It is a Hilbert space for the inner product

$$\langle f_1,f_2\rangle=\int_{\Gamma\backslash G}f_1(g)\overline{f_2(g)}\,d\dot{g}$$

and smooth functions constitute a dense subspace. Moreover, letting G act by right translation, $L^2(\Gamma\backslash G,\chi)$ is a continuous unitary representation of G.

Spectral problem (v.2): decompose $L^2(\Gamma\backslash G,\chi)$ into irreducibles.

References: [**B**, §2.1]. See also Knapp's books for Iwasawa decomposition and the corresponding decomposition of measures.

Week 4

Lecture 9. K-isotypic decomposition of a unitary representation, proof in the case of $\mathrm{SL}(2,\mathbb{R})$, by means of Fejér's kernel.
Admissible representations. Harish-Chandra's Admissibility Theorem: unitary irreducible representations of reductive groups are admissible. Consider the K-isotypic decomposition of $L^2(\Gamma\backslash G,\chi)$:

$$L^2(\Gamma\backslash G,\chi)=\sum_{k\in\mathbb{Z}}^{\oplus}L^2(\Gamma\backslash G,\chi,k)$$

where

$$L^2(\Gamma\backslash G,\chi,k)=\left\{f\in L^2(G)\ ,\ \forall\gamma\in\Gamma,\forall\theta\in\mathbb{R}/2\pi\mathbb{Z}\ ,\ f(\gamma gR_\theta)=\chi(\gamma)e^{ik\theta}f(g)\right\}.$$

The map σ_k defined on $C^\infty(\Gamma\backslash\mathscr{H},\chi,k)$ by

$$\sigma_kf(g)=(f|_kg)\,(i)$$

is an isometric isomorphism of Hilbert spaces:

$$\sigma_k:L^2(\Gamma\backslash\mathscr{H},\chi,k)\xrightarrow{\sim}L^2(\Gamma\backslash G,\chi,k).$$

References: [**B**, §2.1]. See also Katznelson for details about the Fejér Kernel.

Lecture 10. Image of Maass operators under the isomorphisms σ_k:

$$\sigma_{k+2} \circ R_k = R \circ \sigma_k$$

$$\sigma_{k-2} \circ L_k = L \circ \sigma_k$$

$$\sigma_k \circ \Delta_k = \Delta \circ \sigma_k$$

where R, L and Δ are given in the coordinates x, y, θ of the Iwasawa decomposition (1) by:

$$R = iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{1}{2i}\frac{\partial}{\partial \theta}$$

$$L = -iy\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} - \frac{1}{2i}\frac{\partial}{\partial \theta}$$

$$\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + y\frac{\partial^2}{\partial x \partial \theta}.$$

Lie algebras: definition, Lie algebra $\mathrm{Lie}(A)$ associated with an associative algebra A:

$$[a, b] = ab - ba.$$

The Lie algebra of a closed subgroup G of $\mathrm{GL}(n, \mathbb{R})$:

$$\mathfrak{g} = \left\{x \in M_n(\mathbb{R}) \,,\ \forall t \in \mathbb{R}, e^{tx} \in G\right\}.$$

Examples: using the relation $\det(e^x) = e^{\mathrm{Tr}\, x}$, one proves that

$$\mathfrak{so}(n, \mathbb{R}) = \{x \in M_n(\mathbb{R}) \,,\ x^\intercal + x = 0\}$$

$$\mathfrak{sl}(n, \mathbb{R}) = \{x \in M_n(\mathbb{R}) \,,\ \mathrm{Tr}\, x = 0\}$$

$$\mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R}).$$

References: [**B**, §2.2] and P. Garett's notes on **Invariant differential operators**.

Lecture 11. If a Lie group G acts smoothly on the right of a manifold $\mathcal{M}$, then it acts on $C^\infty(\mathcal{M})$ *via*

$$g \cdot f(m) = f(m \cdot g)$$

and $\mathfrak{g}$ acts by the differential operators X_x where

$$X_x f(m) = \left.\frac{d}{dt}\right|_{t=0} f(m \cdot e^{tx}).$$

These two actions do not commute but they satisfy, for $g \in G$ and $x \in \mathfrak{g}$,

$$gX_x g^{-1} = X_{\mathrm{Ad}(g)x}$$

where the adjoint representation $\mathrm{Ad} : G \to \mathrm{End}(\mathfrak{g})$ is defined by $\mathrm{Ad}(g)x = gxg^{-1}$. We admit (for now) the important fact that $x \mapsto X_x$ is a Lie algebra morphism, that is,

$$X_{[x,y]} = X_x X_y - X_y X_x.$$

The universal enveloping algebra: there is an (associative) algebra $\mathcal{U}(\mathfrak{g})$ such that for every algebra A,

$$\mathrm{Hom}_{\mathrm{assoc.}}(\mathcal{U}(\mathfrak{g}), A) = \mathrm{Hom}_{\mathrm{Lie}}(\mathfrak{g}, \mathrm{Lie}(A)).$$

In other words, the functor $\mathcal{U}(-)$ is a left adjoint for $\mathrm{Lie}(-)$.
Construction of $\mathcal{U}(\mathfrak{g})$: consider the ideal I in the tensor algebra $\mathcal{T}(\mathfrak{g})$ generated by elements of the form $x \otimes y - y \otimes x - [x, y]$ and let

$$\mathcal{U}(\mathfrak{g}) = \mathcal{T}(\mathfrak{g})/I.$$

The adjoint action $G \curvearrowright \mathfrak{g}$ extends to an action $G \curvearrowright \mathcal{U}(\mathfrak{g})$ and the map $x \mapsto X_x$ also extends to $\mathcal{U}(\mathfrak{g})$ by the universal property.
Killing form κ, Cartan's criterion for semisimplicity:

$$\kappa(x, y) = 2n \operatorname{Tr}(xy) - 2 \operatorname{Tr}(x) \operatorname{Tr}(y)$$

on $\mathfrak{gl}(n, \mathbb{R})$ (degenerate) and $\mathfrak{sl}(n, \mathbb{R})$ (non-degenerate) so $\mathfrak{sl}(n, \mathbb{R})$ is semisimple, and $\mathfrak{gl}(n, \mathbb{R})$ is not. In addition, κ is G-invariant:

$$\kappa\left(\mathrm{Ad}(g)x, \mathrm{Ad}(g)y\right) = \kappa(x, y)$$

hence defines a G-equivariant identification $\mathfrak{g} \simeq \mathfrak{g}^*$, where G acts on $\mathfrak{g}^*$ *via* the contragredient of Ad. Since $\mathfrak{g}$ is finite-dimensional, one can consider the composition

$$\alpha : \mathrm{End}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{g} \otimes \mathfrak{g}^* \xrightarrow{\kappa} \mathfrak{g} \otimes \mathfrak{g} \longrightarrow \mathcal{T}(\mathfrak{g}) \longrightarrow \mathcal{U}(\mathfrak{g}).$$

The *Casimir element* is

$$\Omega = \alpha(\mathrm{Id}_{\mathfrak{g}}).$$

Since α is G-equivariant, it is an element of $\mathcal{Z}(\mathfrak{g})$, that is, a G-invariant element in $\mathcal{U}(\mathfrak{g})$.

References: [**B**, §2.2] and P. Garett's notes on **Invariant differential operators**. See also S. Sternberg's notes on **Lie algebras**.

Week 5

Lecture 12. Elements in the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$,

$$\mathcal{Z}(\mathfrak{g}) = \{A \in \mathcal{U}(\mathfrak{g}) \ , \ \mathrm{Ad}(G)A = A\}$$

define G-left-invariant differential operators on manifolds of the form G/H.
Case of $\mathrm{SL}(2, \mathbb{R})$: the matrices

$$H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \quad X = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad , \quad Y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

constitute a basis of $\mathfrak{sl}(2, \mathbb{R})$ and satisfy the relations

$$[H, X] = 2X \quad , \quad [H, Y] = -2Y \quad , \quad [X, Y] = 0.$$

Under the identification $\mathfrak{sl}(2, \mathbb{R})^* \simeq \mathfrak{sl}(2, \mathbb{R})$, the dual basis of $\{H, X, Y\}$ is $\{\frac{1}{2}H, Y, X\}$. Therefore, the Casimir element can be expressed as

$$\Omega = \frac{1}{2}H^2 + XY + YX$$

where **products are taken in** $\mathcal{U}(\mathfrak{g})$. Observe that $X - Y \in \mathfrak{so}(2)$ so

$$(X - Y) \cdot f = 0$$

for any SO(2)-invariant function f on $\mathrm{SL}(2,\mathbb{R})$.

References: P. Garett's notes on **Invariant differential operators**.

Lecture 13. The Casimir operator $\Omega \in \mathcal{Z}(\mathfrak{g})$ acts as $2y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$.
KAK and $K\exp(\mathfrak{p})$ (Cartan) decompositions for $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R})$.

References: [**B**, §2.2]. See also Knapp's book.

Lecture 14. Comments on the Cartan motion group $K \ltimes \mathfrak{g}/\mathfrak{k}$ associated with G and the Mackey-Higson-Afgoustidis analogy.
The convolution ring $C_c^\infty(K\backslash G/K)$ of K-bi-invariant functions on G is commutative (Gelfand). If $(G,K) = (\mathrm{SL}(2,\mathbb{R}), \mathrm{SO}(2))$ and $\sigma \in \mathrm{Hom}(K, \mathbb{C}^\times)$, the convolution ring

$$C_c^\infty(K\backslash G/K, \sigma) = \{f \in C_c^\infty(G)\ ,\ f(k_1 g k_2) = \sigma(k_1) f(g) \sigma(k_2)\}$$

is also commutative.
The Spectral Theorem: if T is a compact self-adjoint operators on a Hilbert space $\mathcal{H}$, there exists a Hilbert basis of $\mathcal{H}$ of eigenvectors and the eigenvalues λ_i satisfy $\lim \lambda_i = 0$.

References: [**B**, §2.2].

Week 6

Lecture 15. Compact operators are the limits of finite-rank operators. They form a closed two-sided ideal in $\mathcal{B}(\mathcal{H})$. Hilbert-Schmidt operators: if $K(x,y) \in L^2(X \times X)$, then the operator T defined on $L^2(X)$ by

$$Tf(x) = \int_X K(x,y) f(y)\, dy$$

is compact. Every unitary representation $(\pi, \mathcal{H})$ of G, yields a $*$-representation $\tilde{\pi}$ of the convolution algebra $C_c^\infty(G)$:

$$\tilde{\pi}(\varphi)\xi = \int_G \varphi(g)\pi(g)\xi\, dg$$

satisfies

$$\tilde{\pi}(\varphi_1 * \varphi_2) = \tilde{\pi}(\varphi_1)\tilde{\pi}(\varphi_2) \qquad \text{and} \qquad \tilde{\pi}(\varphi^*) = \tilde{\pi}(\varphi)^*$$

where $\varphi^*(g) = \overline{\varphi(g^{-1})}$. In the case of the right quasi-regular representation ρ on $L^2(\Gamma\backslash G, \chi)$,

$$\rho(\varphi) f(g) = \int_G f(h)\varphi(g^{-1}h)\, dh.$$

This is a Hilbert-Schmidt operator with kernel

$$K(g,h) = \sum_{\gamma \in \Gamma} \chi(\gamma)\varphi(g^{-1}\gamma h).$$

Moreover,

$$\rho(\varphi)\left(L^2(\Gamma\backslash G, \chi)\right) \subset C^\infty(\Gamma\backslash G, \chi)$$

and, if $\varphi(R_\theta g) = e^{-ik\theta}\varphi(g)$, then

$$\rho(\varphi)\left(L^2(\Gamma\backslash G,\chi)\right) \subset C^\infty(\Gamma\backslash G,\chi,k).$$

References: [**B**, §2.3].

Lecture 16. Guest lecture by J. Voight: *On the arithmetic significance of* $\lambda = \frac{1}{4}$.

References: see also [**B**, §Chap. I].

Lecture 17. Let F is a closed G-invariant space of $L^2(\Gamma\backslash G,\chi)$, with K-isotypical decomposition

$$F = \sum_{k\in\mathbb{Z}}^{\oplus} F_k.$$

If $F_k \neq \{0\}$, then Δ has a non-zero eigenvector in $F_k^\infty = F_k \cap C^\infty(\Gamma\backslash G,\chi)$.
The representation $L^2(\Gamma\backslash G,\chi)$ of G is semisimple: it decomposes as the direct sum of unitary irreducible representations of G.

References: [**B**, §2.3].

Week 7

Lecture 18. For $\sigma \in \widehat{\mathrm{SO}(2)}$ and ξ character of $C_c^\infty(K\backslash G/K,\sigma)$, let

$$\mathcal{H}(\xi) = \left\{f \in L^2(\Gamma\backslash G,\chi,k)\ ,\ \rho(\varphi)f = \xi(\varphi)f \quad \text{for all} \quad \varphi \in C_c^\infty(K\backslash G/K,\sigma)\right\}.$$

The spaces $\mathcal{H}(\xi)$ are finite-dimensional, mutually orthogonal and

$$L^2(\Gamma\backslash G,\chi,k) = \sum_{\xi}^{\oplus}\mathcal{H}(\xi).$$

It follows that $L^2(\Gamma\backslash\mathscr{H},\chi,k)$ decomposes as the Hilbert direct sum of eigenspaces for the weight k Laplacian Δ_k. One can also prove that

$$\sum_{\lambda\in\mathrm{Sp}(\Delta_k)}\lambda^{-2}$$

converges, from which it follows that Δ_k has a self-adjoint extension to $L^2(\Gamma\backslash\mathscr{H},\chi,k)$.

References: [**B**, §2.3].

Lecture 19. Construction of smooth vectors: if $(\pi, \mathcal{H})$ is a representation on a Hilbert space and $\xi \in \mathcal{H}$, then $\tilde{\pi}(\varphi)\xi \in \mathcal{H}^\infty$ for $\varphi \in C_c^\infty(G)$. Using a Dirac sequence, it follows that smooth vectors are dense in $\mathcal{H}$.
Overview of the representation theory of compact groups: a locally compact group is compact if and only if it has finite Haar measure, which can be assumed equal to 1.
All representations on Hilbert spaces can unitarized: if $(\pi, \mathcal{H})$ is a representation on a Hilbert space, then π is unitary for the inner product

$$\langle \xi, \eta \rangle = \int_G \langle \pi(g)\xi, \pi(g)\eta \rangle_{\mathcal{H}} \, dg,$$

which defines the same topology.
If $(\pi_1, \mathcal{H}_1)$ and $(\pi_2 \mathcal{H}_2)$ are unitary representations of a compact group G that possess matrix coefficients f_1 and f_2 which are not orthogonal in $L^2(G)$, then there exists a non-trivial intertwiner $L : \mathcal{H}_1 \longrightarrow \mathcal{H}_2$, namely, if $f_i(g) = \langle \pi_i(g)\xi_i, \eta_i \rangle$,

$$\xi_1 \longmapsto \int_G \langle \pi_1(g)\xi_1, \eta_1 \rangle \pi_2(g^{-1})\eta_2 \, dg.$$

Peter-Weyl Theorem: if G is a compact group,

(i) Matrix coefficients of finite dimensional unitary representations are dense in $C(G)$ and $L^p(G)$ for $1 \leq p \leq \infty$;
(ii) Unitary irreducible representations of G are finite-dimensional;
(iii) All unitary representations are semisimple.

In other words,

$$L^2(G) \simeq \sum_{\pi \in \widehat{G}}^{\oplus} V_\pi^* \otimes V_\pi \simeq \sum_{\pi \in \widehat{G}}^{\oplus} \dim(V_\pi) V_\pi.$$

A representation π of a (non-compact) group G with maximal compact subgroup K is said *admissible* if all its K-isotypical components are finite-dimensional. In other words,

$$\pi|_K \simeq \sum_{\rho \in \widehat{K}}^{\oplus} m_\rho V_\rho$$

with all multiplicities m_ρ finite. A famous theorem of Harish-Chandra says that unitary irreducible representations of Lie groups are admissible. We will prove in the case of $G = \mathrm{SL}(2, \mathbb{R})$ that all the representations that occur in $L^2(\Gamma \backslash G, \chi)$ are admissible.

References: [**B**, §2.4].

Lecture 20. If $(\pi, \mathcal{H})$ is a unitary irreducible representation of $G = \mathrm{SL}(2, \mathbb{R})$, then for each $k \in \mathbb{Z} \simeq \widehat{K}$, the isotypical component $\mathcal{H}_k$ is an irreducible $C_c^\infty(K \backslash G / K, \sigma_k)$-module and has dimension at most 1.
Introductory example of $(\mathfrak{g}, K)$-module: trigonometric polynomials in $L^2(\mathbb{T})$. Action of K, action of $\mathfrak{g}$. General definition of $(\mathfrak{g}, K)$-modules.

References: [**B**, §2.4] and Casselman's essays.

Week 8

Lecture 21. K-finite vectors of an admissible representation are smooth and everywhere dense; they form a $(\mathfrak{g}, K)$-module. Representations with isomorphic $(\mathfrak{g}, K)$-modules are said *infinitesimally equivalent.*

References: [**B**, §2.4], see also Knapp.

Lecture 22. The complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{sl}(2,\mathbb{R})$ is generated by

$$R = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \quad , \quad L = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \quad , \quad H = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

subject to the relations

$$[H, R] = 2R \quad , \quad [H, L] = -2L \quad , \quad [R, L] = H.$$

There is also a Casimir element $\Omega \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$ defined by

$$-4\Omega = H^2 + 2RL + 2LR.$$

This element acts by a scalar on every irreducible admissible $(\mathfrak{g}, K)$-module. If V is an irreducible admissible $(\mathfrak{g}, K)$-module and $k \in \mathbb{Z}$, let $V(k)$ denote the isotypical component of V associated with $\sigma_k : R_\theta \mapsto e^{ik\theta}$. Then,

$$V(k) = \{x \in V \ , \ Hx = kx\}$$

$$R : V(k) \longrightarrow V(k+2) \qquad \text{and} \qquad L : V(k) \longrightarrow V(k-2).$$

If $V(k) \ni x \neq 0$, then $\mathbb{C}R^n x = V(k+2n)$, and $\mathbb{C}L^n x = V(k-2n)$ and

$$V = \mathbb{C}x \oplus \bigoplus_{n>0} \mathbb{C}R^n x \oplus \bigoplus_{n>0} \mathbb{C}L^n x.$$

If Ω acts by λ on V, then for $x \in V(k)$

$$LRx = \left(-\lambda - \frac{k}{2}\left(1 + \frac{k}{2}\right)\right) x \qquad \text{and} \qquad RLx = \left(-\lambda + \frac{k}{2}\left(1 - \frac{k}{2}\right)\right) x.$$

If $V(k)$ contains a non-zero vector x such that $Rx = 0$ (resp. $Lx = 0$), then

$$\lambda = -\frac{k}{2}\left(1 + \frac{k}{2}\right) \qquad \left(\text{resp.} \lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right)\right).$$

It follows that all the K-types of a given admissible irreducible $(\mathfrak{g}, K)$-module have the same parity, giving a dichotomy between *even* and *odd* modules.

Uniqueness results:

- If λ is not of the form $\frac{k}{2}\left(1 - \frac{k}{2}\right)$ with k even (resp. odd), then there exists at most one isomorphism class of even (resp. odd) $(\mathfrak{g}, K)$-modules on which Ω acts by λ. The K-types of such a module are all the even (resp. odd) integers.

- If $\lambda = \frac{k}{2}\left(1-\frac{k}{2}\right)$ with $k \in \mathbb{Z}$, then the K-types of an irreducible admissible $(\mathfrak{g}, K)$-module with parity $k \mod 2$ on which Ω acts by λ must be one of the following:

$$\Sigma^+(k) = \{\ell \in \mathbb{Z} \quad , \quad l = k \mod 2 \quad , \quad \ell \geq k\}$$

$$\Sigma^-(k) = \{\ell \in \mathbb{Z} \quad , \quad l = k \mod 2 \quad , \quad \ell \leq -k\}$$

$$\Sigma^0(k) = \{\ell \in \mathbb{Z} \quad , \quad l = k \mod 2 \quad , \quad |\ell| < k\}$$

  and there exists at most one isomorphism class of irreducible admissible $(\mathfrak{g}, K)$-module with a given set of such K-types.

It remains to prove the existence and study the realizability of $(\mathfrak{g}, K)$-modules corresponding to these situations.

References: [**B**, §2.5].

Lecture 23. (Generalized, non-unitary) principal series: for $(\varepsilon, s) \in \{0,1\} \times \mathbb{C}$,

$$H^\infty(\varepsilon, s) = \left\{ f \in C^\infty(G) \ , \ f\left(\begin{bmatrix} u & t \\ 0 & u^{-1} \end{bmatrix} g\right) = [u]^\varepsilon |u|^{\nu+1} f(g) \right\} \subset \mathrm{Ind}_{MAN}^G \sigma_\epsilon \otimes \chi_\nu \otimes 1_N$$

where $s = \frac{\nu+1}{2}$ and $\sigma_\epsilon(m) = m^\varepsilon$ for $m \in \{\pm 1\} \simeq M$, $\chi_\nu(a) = a^\nu$ for $a \in \mathbb{R}^\times \simeq A$ and 1_N is the trivial representation of N. A function in $H^\infty(\varepsilon, s)$ is determined by its restriction to K, which must be even or odd. Conversely, any even or odd function f on $\mathbb{T}$ extends to an element of $H^\infty(\varepsilon, s)$ by

$$\tilde{f}\left(\begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} R_\theta\right) = y^s f(\theta).$$

Complete $H^\infty(\varepsilon, s)$ into a Hilbert space $\mathcal{H}(\varepsilon, s)$ for the norm associated with

$$\langle f_1, f_2 \rangle = \langle f_1|_K, f_2|_K \rangle_{L^2(K)}.$$

Action of $\mathfrak{g}$ on K-finite vectors: $\mathcal{H}(\varepsilon, s)_{(K)}$ is generated by functions of the form

$$f_\ell\left(\begin{bmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{bmatrix} R_\theta\right) = y^s e^{i\ell\theta}$$

which satisfy

$$Hf_\ell = \ell f_\ell \quad , \quad Rf_\ell = \left(s + \frac{\ell}{2}\right) f_{\ell+2} \quad , \quad Lf_\ell = \left(s - \frac{\ell}{2}\right) f_{\ell-2} \quad , \quad \Delta f_\ell = s(1-s) f_\ell.$$

It follows that the irreducible admissible $(\mathfrak{g}, K)$-modules of $\mathrm{SL}(2, \mathbb{R})$ can be realized as subquotients of $\mathcal{H}(\varepsilon, s)_{(K)}$ for some $(\varepsilon, s) \in \{0,1\} \times \mathbb{C}$:

- If $\lambda = s(1-s)$ is not of the form $\frac{k}{2}\left(1-\frac{k}{2}\right)$ with $k = \varepsilon \mod 2$, then $\mathcal{H}(\varepsilon, s)_{(K)}$ is the unique irreducible admissible $(\mathfrak{g}, K)$-module on which Δ acts by λ. Its set of K-types is $2\mathbb{Z} + \varepsilon$. We denote by $\mathcal{P}(\lambda, \varepsilon)$ its isomorphism class and call it *principal series representation.*

- If $\lambda = \frac{k}{2}\left(1 - \frac{k}{2}\right)$ with $1 < k = \varepsilon \mod 2$, there exists three irreducible subquotients of $\mathcal{H}(\varepsilon, s)_{(K)}$ on which Δ acts by λ, with respective sets of K-types $\Sigma^+(k)$, $\Sigma^-(k)$ and $\Sigma^0(k)$. The isomorphism classes corresponding to $\Sigma^\pm(k)$ are denoted by $\mathcal{D}^\pm(k)$ and called *discrete series representations*. The corresponding modules $\mathcal{D}^\pm(1)$ for $k = 1$ are called *limits of the discrete series*.

References: [**B**, §2.5].

Week 9

Lecture 24. Unitarizability of the principal series: if $\lambda \geq \frac{1}{4}$, then $\mathcal{P}(\lambda, \varepsilon)$ contains a unitary representative. Conversely, if $(\pi, \mathcal{H})$ is a unitary admissible representation of $\mathrm{SL}(2, \mathbb{R})$ on which Ω acts by λ, then $\lambda \in \mathbb{R}$. Moreover,

- if $(\pi, \mathcal{H}) \in \mathcal{P}(\lambda, 0)$, then $\lambda > 0$;
- if $(\pi, \mathcal{H}) \in \mathcal{P}(\lambda, 1)$, then $\lambda > \frac{1}{4}$.

This shows that the unitarizable principal series are the $\pi_{\varepsilon,\nu} = \mathrm{Ind}_{MAN}^G \sigma_\epsilon \otimes \chi_\nu \otimes 1_N$ with $\nu \in i\mathbb{R}$ and possibly $\pi_{0,\nu}$ with $-1 < \nu < 1$. These can be shown to be unitarizable, using intertwining integrals. They are called the *complementary series*.
Finite dimensional representations: the only finite-dimensional unitary irreducible representations of $\mathrm{GL}(n, \mathbb{R})^+$ are one-dimensional, of the form $\det^r$ with $r \in i\mathbb{R}$. As a by-product of the proof, $\mathrm{SL}(2, \mathbb{R})$ has no non-trivial finite dimensional unitary irreducible representation.
Unitary irreducible representations that are infinitesimally equivalent, *i.e.* have isomorphic $(\mathfrak{g}, K)$-modules, are unitarily equivalent.

References: [**B**, §2.6].

Lecture 25. Induced representations of finite groups: we consider G finite group, H subgroup of G and V a representation of G. Restricting V to a representation of H gives a functor $\mathrm{Res}_H^G : \mathrm{Rep}(G) \longrightarrow \mathrm{Rep}(H)$.
If $V \in \mathrm{Rep}(G)$ and $W \subset V$ is an H-invariant subspace, then $W \in \mathrm{Rep}(H)$ and for $g \in G$, the space $g \cdot W$ only depends on gH. We say that V is *induced by* W if

$$V = \bigoplus_{\sigma \in G/H} \sigma \cdot W.$$

Example: the left regular representation of G is induced by the left regular representation of H. For every $W \in \mathrm{Rep}(H)$ there exists a unique representation of G induced by W. We denote it by $\mathrm{Ind}_H^G W$.
Example: the regular representation of G is induced by the regular representation of H.
Frobenius Reciprocity is the fact that the functors Ind_H^G and Res_H^G are adjoint to each other:

$$\mathrm{Hom}_H(W, \mathrm{Res}_H^G U) \simeq \mathrm{Hom}_G(\mathrm{Ind}_H^G W, U).$$

Other pictures of induced representations: for $W \in \mathrm{Rep}(H)$, consider

$$\mathrm{Ind}_H^G W = \{f : G \longrightarrow V \ , \ f(gh) = h^{-1} f(g) \quad \text{for all } g \in G, \, h \in H\}$$

with a left action of G by $g \cdot f = f(g^{-1} \cdot)$.

One can also consider $\mathbb{C}[G]$ as a $\mathbb{C}[G]-\mathbb{C}[H]$-bimodule. Then, there is a G-equivariant specialization isomorphism

$$\alpha : \mathbb{C}[G] \otimes_{\mathbb{C}[H]} W \xrightarrow{\sim} \mathrm{Ind}_H^G W$$

defined by

$$\alpha(a,\xi)(g) = \sum_{h\in H} a(gh)\, h\cdot w$$

References: Fulton-Harris.

Lecture 26. Unitarizability of the discrete series: for $k \geq 2$, the infinitesimal class $\mathcal{D}^{\pm}(k)$ admits a unitary representative, namely the space of holomorphic functions f on $\mathscr{H}$ such that

$$\int_{\mathscr{H}} |f(z)|^2 y^k \,\frac{dx\,dy}{y^2} < \infty$$

with $G = \mathrm{SL}(2,\mathbb{R})$ acting by

$$\pi^{\pm}(g)f(z) = (\mp bz + d)^{-k} f\left(\frac{az \mp c}{\mp bz + d}\right).$$

These representations can also be realized as irreducible subrepresentations of the left regular $L^2(G)$.
Solution of the spectral problem: summary of the correspondence between automorphic forms and unitary irreducible representations of $\mathrm{SL}(2,\mathbb{R})$. Holomorphic modular forms occur in the discrete series.

References: [**B**, §2.6, 2.7].

Lecture 27. Abstract and concrete C*-algebras, commutative C*-algebras are algebras of continuous functions (Gelfand isomorphism) and all C*-algebras can be seen as algebras of bounded operators on a Hilbert space.
For G locally compact group, consider the convolution $*$-algebra $C_c(G)$:

$$f * g(s) = \int_G f(t)g(t^{-1}s)\,dt \qquad , \qquad f^*(t) = \Delta_G(t)^{-1}\overline{f(t^{-1})}$$

and equip it with the norm

$$\|f\|_r = \|\tilde{\lambda}_G(f)\|_{\mathrm{op}}$$

where $\tilde{\lambda}_G(f)$ is the operator of convolution by f on the left, acting on $L^2(G)$. More generally, if π is a unitary representation of G, define $\tilde{\pi}(f)$ acting on $\mathcal{H}_\pi$ as in Lecture 15 and consider

$$\|f\|_{\max} = \sup_{\pi} \|\tilde{\pi}(f)\|_{\mathrm{op}}.$$

The completions of $C_c(G)$ with respect to these norms are C*-algebras, respectively denoted by $\mathrm{C}^*_r(G)$ and $\mathrm{C}^*(G)$. The correspondence $\pi \mapsto \tilde{\pi}$ induced a bijection between unitary (resp. tempered) representations of G and non-degenerate representations of $\mathrm{C}^*(G)$ (resp. $\mathrm{C}^*_r(G)$). In other words, the study of unitary representations of G is equivalent to the study of Hilbert spaces that are modules over the C*-algebra(s) of G.

Hilbert C*-modules and bounded adjointable operators. If A and B are C*-algebras, an (A, B)-correspondence is a Hilbert module E over B together with a $*$-morphism

$$\varphi : A \longrightarrow \mathcal{L}_B(E).$$

Given such a bimodule ${}_A E_B$ and a $*$-representation $\mathcal{H}$ of B, one can equip the tensor product $E \otimes_B \mathcal{H}$ with the inner product defined by

$$\langle e_1 \otimes \xi_1, e_2 \otimes \xi_2 \rangle = \langle \xi_1, \langle e_1, e_2 \rangle \xi_2 \rangle.$$

It carries a left action of A *via* φ and the Hilbert completion gives a $*$-representation, denoted $\operatorname{Ind}_B^A \mathcal{H}$.

Week 10

Lecture 28. Mackey induction for locally compact groups: induces unitary representations to unitary representations. Rieffel's construction: if H is a closed subgroup of G, there exists a C*-correspondence

$${}_{\mathrm{C}^*(G)} E(G)_{\mathrm{C}^*(H)}$$

such that for every unitary representation $\mathcal{H}$ of H, there is a specialization isomorphism

$$E(G) \otimes_{\mathrm{C}^*(H)} \mathcal{H} \longrightarrow \operatorname{Ind}_H^G \mathcal{H}$$

that intertwines the left $\mathrm{C}^*(G)$ actions.
If $P = L \ltimes N$ is a parabolic subgroup of a real reductive group G, there exists a $(\mathrm{C}^*_r(G), \mathrm{C}^*_r(L))$-correspondence $\mathcal{E}(G/N)$ that realizes parabolic induction: there is a specialization isomorphism of $\mathrm{C}^*_r(G)$-modules

$$\mathcal{E}(G/N) \otimes_{\mathrm{C}^*_r(L)} \mathcal{H}_{\sigma\otimes\chi} \xrightarrow{\sim} \operatorname{Ind}_P^G \sigma \otimes \chi \otimes 1_N = \pi_{\sigma,\chi}.$$

Adjoint of the functor $\mathcal{E}(G/N) \otimes_{\mathrm{C}^*_r(L)} \cdot$? Case of p-adic groups: Frobenius reciprocity and Bernstein's Second Adjoint Theorem.