Math 13 Fall 2009 Homework 8 Due 11/20/2009

1.) We have seen in class that for a conservative vector field, $\mathbf{F} = \nabla f$, it is the case that curl$\mathbf{F} = (0,0,0)$. Furthermore we also saw that for a vector field $\mathbf{H}$ which is the curl of another vector field, that is $\mathbf{H} = \text{curl}\mathbf{G}$, it is the case that div$\mathbf{H} = 0$. You may wonder if there is a special relationship for a scalar function f which is the divergence of a vector field $\mathbf{X}$, that is $f = \text{div}\mathbf{X}$ (here we are using the symbol $\mathbf{X}$ to represent a vector field). For example, you might believe (but you would be wrong) that maybe $\nabla f = 0$ if $f = \text{div}\mathbf{X}$. In this problem you will show that no such relationship is possible because *every* continuous function $f : \mathbb{R}^3 \to \mathbb{R}$ is the divergence of some vector field $\mathbf{X}$.

a.) To help you get an idea of how to setup your proof, first do the following calculations which finds the vector field $\mathbf{X}$ whose divergence is the function $f(x, y, z) = x^2 + \sin(yz)$. First find the function $g(x, y, z) = \int_0^x f(t, y, z)dt$.

b.) Calculate the divergence of the vector field $\mathbf{G}(x, y, z) = (g(x, y, z), 0, 0)$.

c.) Prove that in general, for any continuous function $f : \mathbb{R}^3 \to \mathbb{R}$, the vector field $\mathbf{G}(x, y, z) = (\int_0^x f(t, y, z)dt, 0, 0)$ has divergence equal to f.

d.) Why is it not the case that if $f = \text{div}\mathbf{X}$, $\nabla f = 0$?

e.) Write a few sentences explaining why there can be no special relationship (or equation) involving a scalar function f which is the divergence of a vector field $\mathbf{X}$. (*Hint*: This is easy.)

2.) Is it possible that two different vector fields $\mathbf{F}$ and $\mathbf{G}$ have div$\mathbf{F} = \text{div}\mathbf{G}$? If not, prove it. If so, give an example where it is true. You may find it helpful to consider your work in problem 1.

3.) Use a computer graphing system to draw one arch of the cycloid $\mathbf{r}(t) = (t - \sin t, 1 - \cos t)$. Use Green's Theorem to find the area under one arch of the cycloid.

4) Let $F(x, y) = \langle y^3 - y, -2x^3 \rangle$. Find the positivly oriented closed simple curve c for which the line integral $\int_c F \circ cds$ is maximal. Then compute the line integral.
Hint: Use Green's theorem, the integral will be maximal when you choose the area so that the function you are integrating is always positive. You can

either use the trigonometric identities
$\cos^2(t) = \frac{1}{2}(1 + \cos(2t))$ and $\sin^2(t) = \frac{1}{2}(1 - \cos(2t))$
or a computer algebra program to solve the integral.