Name ____

Scoring: [do not write here]

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bonus

2nd Midterm– Math 13

The student is reminded that no ancillary aids [e.g. books, notes or guidance from other students] are allowed on this exam. You may use your calculator, but be cautioned that the exam is structured so that you should not need it.

<u>NOTE</u> The purpose of this class is to understand the various types of integration in various types of spaces rather than to test your ability to calculate the integrals in question. So, in general simplify your integrals as much as possible and understand that doing so will garner a good deal of partial credit provided any errors are algebraic in nature.

1[15 points]. Pick Your Parametric Poison: Pick any 3 of the following 4 surfaces and parameterize it in the most natural way you can.

a) The triangle whose vertices are $(0,0,2),(1,2,4),(2,4,7)$.

We first subtract one of the points from the other two. To parameterize the triangle whose vertices are $(0,0,0),(1,2,2),(2,4,5)$ we use $\langle u,v\rangle \to u\langle 1,2,2\rangle + v\langle 2,4,5\rangle = \langle u+2v, 2u+4v, 2u+5v\rangle, \quad 0 \leq u \leq 1, 0 \leq v \leq u$

b) The part of the cylinder $x^2+y^2=3$ in between the planes $z=-x-y$ and $z=x-y+1$

The cylinder is parameterized by $\langle u,v\rangle \to \langle \sqrt{3}cos(v), \sqrt{3}sin(v), u\rangle$. we need $u=z$ to be between $-x-y$ and $x-y$, so we simply require $-cos(v)-sin(v) \leq u \leq -cos(v)+sin(v)$ for those regions where $sin(v) \geq 0$ and the opposite $-cos(v)+sin(v) \leq u \leq -cos(v)-sin(v)$ when $sin(v) < 0$. You will receive FULL credit if you just had the first and bonus if you realized that it needed to be broken apart.

c) The region under the graph of $z=x^2y^2$ along the line $y=x^2$.

A cylinder is like a circle moved up and down by allowing z to be free, this surface is the same except you are moving up and down the graph $y=x^2$ and the upper limit is $z=x^2y^2$. No lower limit is given, if you assumed I meant only the region above the xy-plane, that is fine [like when we say the area "under the graph" when we really mean "under the graph and above the x-axis"].

So, the parametrization is $\langle u,v\rangle \to \langle u,u^2,v\rangle, 0 \leq v \leq u^6$. No bounds are given for v as there are none given in the problem.

d) The surface of revolution of $z=x$ rotated about the z axis for $0 \leq x \leq 5$.

$$\langle u, v\rangle \rightarrow \langle u \cdot cos(v), u \cdot sin(v), u\rangle$$

2 [20 points]. The current in a particular region of the ocean is given by $\mathbf{F}(x, y, z) = \langle -y, x, z \rangle$. This vector represents the velocity of the water [it has units of $\frac{m}{s}$].

a) Calculate the flux through a square whose corners are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 0, 1)$, and $(1, 0, 1)$

We first parameterize the square: $G(u, v) = \langle u, 0, v \rangle, 0 \leq u \leq 1, 0 \leq v \leq 1$.

$G_u = \langle 1, 0, 0 \rangle, G_v = \langle 0, 0, 1 \rangle$.

$G_u \times G_v = \langle 0, -1, 0 \rangle$.

$\mathbf{F}(x, y, z) = \langle -y, x, z \rangle = \langle 0, u, v \rangle$. So...

$$\iint_{square} \mathbf{F} \cdot \mathbf{dS} = \int_0^1 \int_0^1 \langle 0, u, v \rangle \cdot \langle 0, -1, 0 \rangle du dv =$$
$$\int_0^1 -u du dv = -\frac{1}{2}$$

b) Based on your answer, is the orientation of your parametrization aligned with the current or against it?

My surface is oriented in a way that is against the current because my answer is negative.

3. [15 points]. A parabolic bowl has a surface obeying the equation $z = \frac{1}{4}(x^2 + y^2)$ for $z \leq 4$. The surface density of the bowl is $f(x, y, z) = \frac{1}{\sqrt{z+1}}$ (this is so that the bottom is denser than the top, giving the bowl added stability).

a) Find the mass of the bottom 1/2 of the bowl [that is $z \leq 2$].

We parameterize the bowl: $G(u, v) = \langle u \cdot cos(v), u \cdot sin(v), \frac{1}{4}u^2\rangle$. $G_u = \langle cos(v), sin(v), \frac{u}{2}\rangle, 0 \leq u \leq 4, 0 \leq v \leq 2\pi$. $G_v = \langle -u \cdot sin(v), u \cdot cos(v), 0\rangle$. This gives $G_u \times G_v = \langle \frac{-u^2}{2} cos(v), \frac{-u^2}{2} sin(v), u\rangle$, the magnitude of which is $\sqrt{\frac{u^4}{4} + u^2} = u\sqrt{1 + \frac{u^2}{4}}$.

We want the mass of the cup below $z = 2$, which corresponds to $u = \sqrt{8}$ Plugging into the template for a surface integral we get:

$$mass = \iint_{cup} f(x, y, z)dS = \int_0^{\sqrt{8}} \int_0^{2\pi} f(G(u, v))|G_u \times G_v \mid dvdu =$$

$$\int_0^{\sqrt{8}} \int_0^{2\pi} \frac{1}{\sqrt{\frac{u^2}{4} + 1}} (u\sqrt{1 + \frac{u^2}{4}})dvdu =$$

$$\int_0^{\sqrt{8}} \int_0^{2\pi} udvdu =$$

$$8\pi$$

b) The mass of the entire bowl happens to twice the mass you got in answer a. This may be surprising since the bottom is denser than the top. How is this possible?

Density is less, but there is more surface area, they balance out.

4[15 points].

In a dart game, points nearer the center are more likely to be hit than points further away. The dartboard has radius 1 and is centered at the origin.

Consider the three following possible values for $f(x,y)$:

(1) $f(x,y) = \frac{3}{2\pi}\sqrt{x^2+y^2}$
(2) $f(x,y) = \frac{1}{4\pi\sqrt{x^2+y^2}}$
(3) $f(x,y) = \frac{1}{(\pi ln(2))(1+x^2+y^2)}$

Only 1 of the above functions is appropriate to be a probability density function for the likelihood of a player hitting a particular region of the dartboard. Which one is it? You must justify your answer.

The first is impossible because we are told that it is more likely for darts to be near the center [origin] than away...so the probability density should be greater there.

The second looks fishy because it isn't even defined at $(0,0)$, but we should check it anyway:

Let $G(u,v) = \langle u \cdot cos(v), u \cdot sin(v)\rangle$ be the parametrization of the dart board. To figure out the distortion we pretend it is in 3=space and then $G_u = \langle cos(v), sin(v), 0\rangle, G_v = \langle -u \cdot sin(v), u \cdot cos(v), 0\rangle$ and $G_u \times G_v = \langle 0, 0, u$, which has magnitude u.

Then we integrate the density function $\int_0^1\int_0^{2\pi}\frac{1}{4\pi u}ududv = \frac{1}{2}$. Since we know that it must hit the dartboard somewhere, this can't be right.

The answer must be the last one. [Let's check]

$$\int_0^1\int_0^{2\pi}\frac{u}{(\pi)(ln(2))(1+u^2)}dvdu =$$
$$\int_0^1\frac{2u}{(ln(2))(1+u^2)}du =$$

Using $w = 1+u^2$ and substitution gives...

$$\frac{1}{ln(2)}(ln(1+u^2)) \mid_0^1 = 1$$

So this is a valid probability density function, appropriate to the dartboard game.

5. [10 points] Earlier in life you were told that the surface area of a sphere is 4π times the radius squared. Show this using surface integrals.

We fix a radius R and parameterize the sphere

$$G(u, v) = \langle(Rcos(u)sin(v), Rsin(u)sin(v), Rcos(v)\rangle$$

$$G_u = \langle -Rsin(u)sin(v), Rcos(u)sin(v), 0\rangle$$

$$G_v = \langle Rcos(u)cos(v), Rsin(u)cos(v), -Rsin(v)\rangle$$

$$G_u \times G_v = \langle -R^2cos(u)sin^2(v), -R^2sin(u)sin^2(v), -R^2sin(v)cos(v)\rangle$$

After a bunch of trig simplification:

$|G_u \times G_v| = R^2 \mid sin(v) \mid$.

To get surface area we do the surface integral of 1 and get $\int_0^{2\pi}\int_0^{\pi} R^2 \mid sin(v) \mid dudv = 4\pi R^2$.

6. [25 points] The altitude in a given region is given by

$$height(x,y) = 1 + x^2 + \frac{y^2}{4}$$

The region is bounded by the ellipse $x^2 + \frac{y^2}{4} = 1$. Zeus is unhappy with this region in particular, and he hurls a lightning bolt at it once per day. The probability density for the location of the lightning strike is directly proportional to the square root of the height...that is $f(x,y) = k\sqrt{(h(x,y))}$. what is the probability that lightning will strike somewhere in the region $x^2+\frac{y^2}{4} \leq 1/4$? NOTE: you may leave your answer in terms of k. You do not need to find k!!!

We are given the probability density function and asked to compute a probability from it, which means we simply integrate the function over the region.

We use the parametrization of an elliptical disk:

$G(u,v) = \langle u \cdot cos(v), 2u \cdot sin(v)\rangle$ To get the stretching factor we pretend it is in three space and get $G_u = \langle cos(v), 2sin(v), 0\rangle$, $G_v = \langle -u \cdot sin(v), 2u \cdot cos(v), 0\rangle$. From which we get that $G_u \times G_v = \langle 0, 0, 2u$, the magnitude of which is $2u$. So to find the probability we calculate the integral:

$$\int_0^{1/2}\int_0^{2\pi} k\sqrt{1 + u^2 \cdot cos^2(v) + \frac{1}{4}4u^2 \cdot sin^2(v)}2udvdu =$$

$$\int_0^{1/2} 2\pi k\sqrt{1+u^2}2udu =$$

Using a substitution with $w = 1 + u^2$ we find this is:

$$\frac{4}{3}\pi k(1+u^2)^{3/2}\Big|_0^{\frac{1}{2}} = \frac{4\pi((\frac{5}{4})^{3/2} - 1)}{3}$$

NOTE

The following two problems are bonus questions. You should only attempt them if you are done with the rest of the exam.

B1. Parameterize a doughnut such that the inner hole has radius 1 and radius of the doughnut itself is $1/2$. What is the surface area?

B2. The probability that I pass my exam can be calculated by integrating a probability density function dependent on how long I studied the night before and how much sleep I got the night before. In particular $f(a,b) = \frac{a+10}{480}$ for $0 \leq a \leq 4$ and $0 \leq b \leq 10$, where a is the amount of studying you do, and b is the amount of sleep you get. [In this case the probability that you pass after getting exactly x hours of sleep and studying for exactly y hours is $\int_0^x \int_0^y f(a,b) da db$.] Assume that the amount of sleep you get and the amount of studying you do is completely random [Which works out to mean that the probability that you studied somewhere between c and d hours and slept between x and y hours is $\frac{(y-x)(d-c)}{40}$ for $0 \leq c \leq d \leq 4$ and $0 \leq x \leq y \leq 10$].

Given that you got at least 5 hours of sleep and studied for at least 2 hours, what's the likelihood you pass the exam?