Name ____

FINAL- Math 13

The student is reminded that no ancillary aids [e.g. books, notes or guidance from other students] is allowed on this exam. Also, no calculators are allowed on this exam.

1) 25

2) 7

3) 10

4) 15

5) 16

6) 15

7) 4

8) 8

9) 5

10) 4

11) 8

12) 6

13) 12

14) 10

15) 10

16) 45

Section I: NOTE!!!!: If the question asks for an integral, you do not need to actually do the integral. Simply write down what the integral would be [An example of what your answer might look like is $\int\int 3uvdu\ dv$ with appropriate limits]. You must get your integral into a form that has $du\ dv$ [or an analogous term] in it. Writing $\int f\ ds$ gets you nothing.

1. [**][25 points] Let $G(u,v,w) = \langle 3u, 2usin(v), wu\rangle, 0 \leq v \leq \pi, 0 \leq w \leq 1, 0 \leq u \leq 1-w$ be a parametrization of a volume V. What is the volume integral of $f(x,y,z) = x$ on this region?

ANSWER: We must find the proper "finagling" factor for this volume integral. This the Jacobian: the determinant of

$$\begin{bmatrix} \frac{\delta f_1}{\delta x} & \frac{\delta f_1}{\delta y} & \frac{\delta f_1}{\delta z} \\ \frac{\delta f_2}{\delta x} & \frac{\delta f_2}{\delta y} & \frac{\delta f_2}{\delta z} \\ \frac{\delta f_3}{\delta x} & \frac{\delta f_3}{\delta y} & \frac{\delta f_3}{\delta z} \end{bmatrix} = \begin{vmatrix} 3 & 0 & 0 \\ 2\ sin(v) & 2\ u\ cos(v)0 & \\ w & 0 & u \end{vmatrix} = 6u^2cos(v).$$

Thus the volume integral should be $\int_0^\pi \int_0^1 \int_0^{1-w} 3u{\cdot}6u^2cos(v)dudwdv$

2. [*] [7 points] a) Let $\mathbf{r_1(t)}, \mathbf{r_2(t)}, \mathbf{r_3(t)}$ be three curves in space. What is $\frac{d}{dt}$ of $\mathbf{r_1} \cdot (\mathbf{r_2} \times \mathbf{r_3})$?

<u>ANSWER</u>

We use the differentiation rules for differentiation for vector products:

$\frac{d}{dt}\mathbf{r_1} \cdot (\mathbf{r_2} \times \mathbf{r_3}) = \mathbf{r_1'} \cdot (\mathbf{r_2} \times \mathbf{r_3}) + \mathbf{r_1} \cdot (\mathbf{r_2} \times \mathbf{r_3})' = \mathbf{r_1'} \cdot (\mathbf{r_2} \times \mathbf{r_3}) + \mathbf{r_1} \cdot (\mathbf{r_2'} \times \mathbf{r_3} + \mathbf{r_2} \times \mathbf{r_3'})$

3. [*] [10 points] $\mathbf{F} = \nabla g$ where $g(x, y, z) = xyz$. What is $\nabla \times \mathbf{F}$? What is the line integral of $\mathbf{F}$ evaluated along a path that begins at $\langle 1, 2, 3\rangle$ and ends at $\langle 2, 3, 4\rangle$?

ANSWER

$\nabla \times \mathbf{F} = \nabla \times \nabla g = \langle 0, 0, 0\rangle$ since the curl of a grad is always $\hat{0}$. We are told straight out that $\mathbf{F} = \nabla g$ so the line integral of any path from $\langle 1, 2, 3\rangle$ to $\langle 2, 3, 4\rangle$ is, by FToLI, $xyz|_{\langle 2,3,4\rangle} - xyz_{\langle 1,2,3\rangle} = 24 - 6 = 18$.

4. [**] [15 points] $\mathbf{F}$ is the function $\mathbf{F}(\mathbf{u}, \mathbf{v}) = \langle v, uv + 1\rangle$. $\mathbf{G}$ is the function $\mathbf{F}^2$. That is $\mathbf{G}(u, v) = \mathbf{F}(\mathbf{F}(\mathbf{u}, \mathbf{v}))$.

a) Use the chain rule to find the derivative of $\mathbf{G}$ at the point $\langle 2, 1\rangle$.

ANSWER $G = F \circ F$, so by the chain rule $DG|_{\hat{x}} = DF|_{F(\hat{x})} \cdot_{matrix} DF_{\hat{x}}$. In this case $DF|_{x,y} = \begin{bmatrix} 0 & 1 \\ v & u \end{bmatrix}$. Furthermore, $F(2, 1) = \langle 1, 3\rangle$, so The answer is $\begin{bmatrix} 0 & 1 \\ 3 & 1 \end{bmatrix} \cdot_{matrix} \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix}$

b) Is F locally invertible at the point $\langle 2, 3 \rangle$? Justify your answer.

ANSWER

At $\langle 2, 3 \rangle$ the derivative matrix does not have zero determinant, so the function is invertible.

5. [***][16 points] Let $\mathbf{F} = \langle y - x^x, y^y - x, 13z \rangle$. What is the line integral of $\mathbf{F}$ along the path that borders the region above the x-axis and below the function $y = 4 - x^2$ A drawing is provided:

<u>ANSWER</u> The function is too difficult to integrate directly, so we use Stokes' theorem to say $\int_{C_1} \mathbf{F} \cdot dr = \iint_S \nabla \times \mathbf{F} \cdot dS$, where S is the enclosed area. In this case we have $\nabla \times F = \langle 0, 0, -2 \rangle$. We can parameterize the surface as $G(u, v) = \langle u, v \rangle, -2 \leq u \leq 2, 0 \leq v \leq 4 - u^2$. Since the curl is perpendicular to the x y plane, the line integral should simply be -2 times the area of the plane:

$\int_C = \int_{-2}^{2} \int_0^{4-u^2} -2du\ dv$

6. [**][15 points] Let $G(u,v) = \langle 2u+v, u, 2v\rangle, 0 \leq u \leq 2, 0 \leq v \leq 1$ be a parametrization of a surface . What is the surface integral of $f(x,y,z) = (x-2y)z$ on S?

ANSWER

We must find the finagling factor for a surface integral. $G_u = \langle 2,1,0\rangle, G_v = \langle 1,0,2\rangle$. We then have $G_u \times G_v = \langle 2,-4,-1\rangle$. The magnitude of this is $\sqrt{21}$, so the surface integral is $\int_0^2 \int_0^1 (2u+v-2u)(4v)\sqrt{21} dv\ du$

Section 2: Conceptual understanding.

You may omit as many of these as you wish. You will receive half credit for any problems omitted. Please make clear those problems you wish to omit, as writing down a wrong answer may result in your receiving no credit at all.

7. [*][4 points] A particle is moving without stopping along a path described by $\mathbf{r}(\mathbf{t})$. At $t = 1$ we have that $r(t) = \langle 2, -1, 3 \rangle$. If the particle stays a constant length from the origin, what is a possible value for $\mathbf{r}'(\mathbf{t})$. [Note: There are several possible answers, just give me one].

ANSWER

By Bucky's theorem, we must have $\mathbf{r}'(\mathbf{t}) \cdot \mathbf{r}(\mathbf{t}) = 0$ to keep the magnitude constant. HOWEVER we know the particle is moving without stopping, so we can't simply choose $\langle 0, 0, 0 \rangle$. Thus any other vector such that $r'(t) \cdot r(t) = 0$ will work. For example $\langle 1, 2, 0 \rangle$.

8. [*][8 points] $\mathbf{r_1}, \mathbf{r_2}$ are curves in space. What is $\mathbf{r_1} \cdot (\mathbf{r_2} \times \mathbf{r_1}))$? Justify your answer.

<u>ANSWER</u>

Since $r_1 \times r_2$ is perpendicular to r_1 we must have that $r_1 \cdot (r_2 \times r_1) = 0$.

9. [*][5 points] How can you tell if a vector function **F** a conservative vector field? What does it mean to be due to a conservative field?

<u>ANSWER</u>

To tell if a function is a conservative vector field, it suffices to show that $\nabla \times \mathbf{F} = \langle 0, 0, 0 \rangle$ at all points. If this is true we get the following:

- The value of line integrals are independent of path.
- The line integral of any loop is 0.
- There exists some g such that $\mathbf{F} = \nabla g$.

10. [*][4 points] Parameterize the surface of $z = x^2y$ lying within the cylinder $\frac{x^2}{4} + \frac{y^2}{6} = 1$.

ANSWER

$\langle u, v\rangle \rightarrow \langle 2cos(u), \sqrt{6}sin(u)\rangle$ covers the disk trapped inside the cylinder, to parameterize the graph of this disk by $z = x^2y$ we simply tack on the the third component:

$\langle u, v\rangle \rightarrow \langle 2cos(u), \sqrt{6}sin(u), 4\sqrt{6}cos^2(u)sin(u)\rangle$.

11. [**][8 points] Parameterize the following [Remember to include limits]:

a) A triangle with end points $\hat{a}$, $\hat{b}$, $\hat{c}$.

ANSWER

$\langle u, v\rangle \to \hat{a} + u(\hat{b} - \hat{a}) + u(\hat{c} - \hat{a}), 0 \leq u \leq 1, 0 \leq v \leq 1 - u.$

b) The ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$ in the clockwise direction.

ANSWER

$r(t) = \langle \sqrt{a} cos(t), -\sqrt{b} sin(t)\rangle$ [the $-$ is to make it go in the clockwise direction....you could also use $\langle sine, cosine\rangle$ to do the same thing.

c) The filled in parallelepiped such that $\hat{a}$ is one of the points, and the edges of the parallelepiped coming off $\hat{a}$ correspond to the vectors $\hat{b}, \hat{c}, \hat{d}$.

ANSWER
$\langle u, v, w\rangle = \hat{a} + u\hat{b} + v\hat{c} + w\hat{d}, 0 \leq u \leq 1, 0 \leq v \leq 1, 0 \leq w \leq 1.$

d) A circular shape such that the radius at any given point is $f(\theta)$, where θ is the angle made with the x axis.

answer
$\mathbf{r(t) = \langle f(t)cos(t), f(t)sin(t)\rangle}$

12. [**][6 points] **F** is defined everywhere but the "lattice points." [A lattice point is a point where all the co-ordinates are integers...like $\langle 2,3,6\rangle$ is a lattice point, because 2,3,6 are all integers.] The curl of **F** is $\langle 0,0,0\rangle$ everywhere it is defined. Is it possible that there is some circle with radius $\frac{1}{2}$ such that the line integral of **F** on the circle is not 0? [We only consider circles that do not touch lattice points]

ANSWER

No. It is not possible. For any circle you can find some surface that has no poles on it whose boundary is that circle...so even if you had a circle around a lattice point, you could still find a surface [a dome of some kind] that had no poles on it, and the line integral around this circle would be the surface integral of the curl, which would be 0.

Section III: Advanced topics.

13. [**][12 points] **F** is a vector function such that **F** is defined everywhere, and the surface integral of **F** on the surface $\mathbf{G_1}(u,v) = \langle u\, cos(v), u\, sin(v), 0\rangle, 0 \le u \le 1, 0 \le v \le 2\pi$ is 36π. The surface integral of **F** on the surface $\mathbf{G_2}(u,v) = \langle u\ sin(v), u\ cos(v), \sqrt{1-u^2}\rangle 0 \le v \le 2\pi, 0 \le u \le 1$ is 15π. What is the average value of the Divergence of F on the upper unit [filled in] hemisphere. JUSTIFY YOUR ANSWER!!! [note that G_1 is a unit disk and G_2 is a unit hemisphere without its bottom.]

ANSWER

Average value of a function $f = \frac{\int\int\int f dV}{\int\int\int 1 dV}$. By stokes theorem we have that $\int\int\int \nabla \cdot F dV$ should be the negative of the integrals of F because the surfaces are oriented inward rather than outward. Thus we get the top term of the fraction is -51π. The bottom is just the volume, which is $2/3\pi$. So the average value is $\frac{-153}{2}$.

14. [***] [10 points] $\mathbf{F} = \langle y, x, z \rangle$. Use one of the techniques shown in class to find a g such that $\mathbf{F} = \nabla g$.

ANSWER.

I will use the "build up" technique, you could also do the "easy path" technique:

$g_x = y \to g(x, y, z) = xy + h(y, z)$

$g_y = x \to (xy + h(y, z))|_y = x + h_y = x \to h_y = 0 \to h(y, z) = q(z)$

$g_z = z \to (xy + q(z))_z = z \to q_z = z \to q(z) = z^2/2$

So $g(x, y, z) = xy + \frac{z^2}{2}$ works.

15. [**] [10 points] $r(t) = \langle 3t, 4t, 5t \rangle, 0 \leq t \leq 10$. If $s(t)$ is the total distance travelled along the curve from its starting point to the point $r(t)$, what is $\frac{ds}{dt}$? reparameterize the curve so that position is in terms of arc length.

ANSWER

$\frac{ds}{dt} = |r'(t)| = |\langle 3, 4, 5 \rangle = 5\sqrt{2}$.

The "cookbook" way of reparameterizing in terms of arc length is to say $s(T) = \int_0^T |r'(t)| ds = \int_0^T 5\sqrt{2} ds = 5T\sqrt{2}$

So $T = \frac{s}{5\sqrt{2}}$, and $r(s) = \frac{3}{5\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$.

Section IV. Synthesis. <u>NOTE</u> If you do not feel confident in your ability to do this problem, you may do the first "optional problem" instead for 25 points credit. If you do not feel you can do that problem, you may do optional problem 3 for 15 points of credit. If you choose either of these, please write "omit" clearly after the problem statement.

16. [***********] [45 points] F is a vector function about which the following is known:

- **F is defined everywhere except $\langle 0, 1, z\rangle$ for all z [that is to say that there is a "line" missing from the domain...any point such that the x coordinate is 0 and the y-coordinate is 1 is a point at which F is not defined.**
- **Everywhere in the x, y plane F is defined, the curl of $\mathbf{F} = \langle 0, 0, \frac{k}{\sqrt{x^2+y^2}}\rangle$, where k is some constant.**
- **along the x axis, we know $\mathbf{F}(x, 0, 0) = \langle 3\pi x^2, 0, 0\rangle$**
- **if $\mathbf{r(t)} = \langle \frac{1}{2}cos(t), \frac{1}{2}sin(t), 0\rangle, 0 \leq t \leq \pi$ is the upper semi-circle of radius $\frac{1}{2}$, the line integral along r of F equals 2π.**
- **If $\mathbf{r(t)} = \langle 2cos(t), 2sin(t), 0\rangle, 0 \leq t \leq \pi$ is the upper semicircle of radius 2, the line integral along r is 50π.**

Find k and determine the contribution to the line integral due to the enclosed pole.

<u>ANSWER</u>

I am not going to post the complete solution here, but here is the outline:

There is no pole inside the path that borders the semicircle of radius 1/2, so we have $\int\int_S \nabla \times F \cdot dS = \int_{C_1} F \cdot dr + \int_{C_2} \mathbf{F} \cdot dr$, where S is the region inside the semicircle, C_1 is the curve going along the circular part and C_2 is the integral along the "flat" part [along the x axis].

This gives us an equation from which we can solve for k.

Then we have that $\int\int_S = \int_{C_1} \mathbf{F}\cdot dr + \int_{C_2} \mathbf{F}\cdot dr +$Contribution due to pole. Where C_1 is now the circular part of the curve for a semicircle of radius 2 and C_2 is the line segment from

$\langle -2,0,0\rangle$ **to** $\langle 2,0,0\rangle$**. This equation [when you do the integrals] allows you to find the contribution due to the pole.**

OPTIONAL PROBLEMS

O1. $\mathbf{F}(x, y, z) = \langle ln(y^2 + 1), y^2 + x^2, (z - 1)y \rangle$. What is the flux through the cylinder whose defining equation is $x^2 + y^2 = 1, 0 \leq z \leq 2$? [The cylinder has no lids].

O2. Pretend you are in $\Re^4$, so that your variable names are x_1, x_2, x_3, and x_4. Let $\omega = (2x_1 - 3x_2)dx_1 \wedge dx_3 \wedge dx_4$ be a differential form. What is $d\omega$? What is integral of ω on the sides of a hypercube with side length 2 centered at the origin. [Note, the corners of such a cube would be like $\langle -1, 0, 0, 1\rangle, \langle 1, 0, 1, 0\rangle$, etc.].

O3. The probability that Sue arrives by a certain time t is given by a probability density function $f(t) = 2 - kt^2$ [t is in hours]. Sue is willing to wait up to 20 minutes for Roxy, and she may arrive as late as $t = 1$. The probability that Roxy arrives by a certain time t is also given by a probability function, $g(x) = 3t^2$. Roxy also arrives within 1 hour.

Right an integral representing the likelihood that Roxy and Sue meet.

What is k?