Math 16 - Practice Final

May 28th 2007

Name: ______________________________

Instructions

1. In Part A, attempt every question. In Part B, attempt **two** of the five questions. If you attempt more you will only receive credit for your best two answers.
2. Show all your work.
3. No books, notes or calculators allowed.
4. Ask me if something is unclear.

	Question	Possible	Earned
Part A	1	12	
	2	12	
	3	12	
	4	12	
	5	12	
Part B		20	
		20	
Total		100	

Part A

1. (a) (6 pts) What is a LP problem in standard form? Explain how to convert a problem in standard form into a dictionary.

   (b) (6 pts) When solving a problem using the primal simplex method, how do we know when we have reached an optimal dictionary? Why does this guarantee the current solution is optimal?

2. (a) (2 pts) Define a degenerate dictionary.

   (b) (2 pts) Define a degenerate pivot.

   (c) (4 pts) What is Bland's Rule? Why is it useful?

   (d) (4 pts) State the Fundamental Theorem of Linear Programming.

3. In applying the dual simplex method to a LP problem we have reached the following dictionary.

$$\begin{array}{rcrcrcrcr}
\zeta & = & 30 & - & 3x_3 & - & 4x_2 & - & 3w_1 \\
\hline
x_1 & = & -10 & + & 3x_3 & + & 2x_2 & + & w_1 \\
w_3 & = & -21 & + & 3x_3 & + & 6x_2 & +3w_1 & \\
w_2 & = & 13 & - & 2x_3 & - & 3x_2 & - & w_1
\end{array}$$

   (a) (4 pts) Deduce an upper bound on the optimal value of ζ. Justify your answer.

   (b) (8 pts) Perform one iteration of the dual simplex method, using the largest coefficient rule to determine the pivot variables.

4. Consider the following linear program:

$$\begin{array}{lrcrcrcr}
\text{maximise} & x_1 & - & x_2 & + & 3x_3 & & \\
\text{subject to} & & & x_2 & - & 2x_3 & \leq & -1 \\
 & -x_1 & & & + & 2x_3 & \leq & 1 \\
 & 2x_1 & - & 2x_2 & & & \leq & -3 \\
 & & & x_1, \ldots, x_3 & \geq & 0 & &
\end{array}$$

   (a) (4 pts) Write down this problem in matrix notation.

   (b) (4 pts) Write down the dual of this problem.

   (c) (4 pts) Convert the dual into standard form. What do you notice?

5. (a) (2 pts) What is the LP relaxation of the following integer programming problem?

$$
\begin{array}{lrcrcrcrcrcrcrcl}
\text{maximise} & 3x_1 & + & 7x_2 & + & x_3 & + & x_4 & + & 3x_5 & + & 2x_6 & + & x_7 & & \\
\text{subject to} & -x_1 & & & + & x_3 & & & + & x_5 & & & & & = & -7 \\
& x_1 & - & x_2 & & & & & & & & & + & x_7 & = & 2 \\
& & & & - & x_3 & + & x_4 & & & & & & & = & 0 \\
& & & x_2 & & & - & x_4 & - & x_5 & - & x_6 & & & = & 10 \\
& & & & & & & & & & & x_6 & - & x_7 & = & -5 \\
& & & & & & & & & & & x_1, & \ldots, & x_7 & \geq & 0 \\
& & & & & & & & & & & x_1, & \ldots, & x_7 & \text{integers} &
\end{array}
$$

(b) (6 pts) Convert your answer to (a) into a minimum cost network flow problem. Draw the corresponding network.

(c) (4 pts) Explain how you could solve the original problem without using any tools from the theory of integer programming. (You do not need to solve the problem.)

Part B

6. (a) (3 pts) What is a polyhedron?

(b) (5 pts) Explain the connection between polyhedra and LP.

(c) (5 pts) What is the particular significance of vertices of a polyhedron?

(d) (7 pts) Suppose an iteration of the simplex method takes you from the solution $(x_1, x_2) = (0, 1)$ to the solution $(x_1, x_2) = (4, 3)$. Can you deduce one of the constraints of the problem?

7. Consider the LP problem, which has a constant a.

$$
\begin{array}{lrcrcrcl}
\text{maximise} & 4x_1 & - & 22x_2 & - & 4x_3 & & \\
\text{subject to} & -2x_1 & + & 11x_2 & + & 3x_3 & \leq & 1 \\
& x_1 & - & 5x_2 & & & \leq & 1 \\
& 3x_1 & - & 14x_2 & - & 2x_3 & \leq & a \\
& & & & & x_1, \ldots, x_3 & \geq & 0
\end{array}
$$

This has the following dictionary:

$$
\begin{array}{rcrcrcrcr}
\zeta & = & 4 & - & 4x_5 & - & 2x_2 & - & 4x_3 \\
\hline
x_4 & = & 3 & - & 2x_5 & - & x_2 & - & 3x_3 \\
x_1 & = & 1 & - & x_5 & + & 5x_2 & & \\
x_6 & = & \_\_ & + & 3x_5 & - & x_2 & + & 2x_3
\end{array}
$$

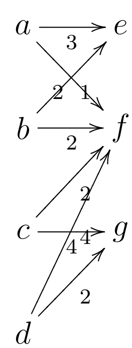
(a) (6 pts) What is the missing value in the third constraint?

(b) (2 pts) For which values of a is this dictionary optimal?

(c) (12 pts) Solve the problem for $a = 1$.

8. (a) (4 pts) Among minimunm cost network flow problems, what characterises the transportation problems?

   (b) Consider the following problem which has supply of 3 at each of nodes a, b, c and d, and demand of 4 at each of nodes e, f and g.



   i. (6 pts) What is the solution in which the following arcs are basic: $(a, e), (b, e), (b, f), (c, f), (d, f), (d, g)$?

   ii. (10pts) Solve the problem using the simplex method.

9. Two players A and B each pick a number from 1 to 4 inclusive. The player with the lower number wins \$1 from his opponent, unless his number is 1 less than his opponent's number, in which case his opponent wins \$1.

   (a) (5 pts) Write down the payoff matrix for this game in which player A is the row player.

(b) (4 pts) Explain what an optimal strategy for a player in a matrix game looks like.

(c) (6 pts) Write down the associated LP problem for player B.

(d) (5 pts) What is the value of this game? (Hint: You should not have to solve an LP problem for this.)

10. (20 pts) Consider the following integer programming problem:

$$\begin{array}{lrcrcrcr}
\text{maximise} & x_1 & + & 2x_2 & + & 3x_3 & & \\
\text{subject to} & x_1 & + & x_2 & + & x_3 & \leq & 10 \\
 & 2x_1 & - & x_2 & & & \leq & 4 \\
 & x_1 & - & x_2 & + & 4x_3 & \leq & 8 \\
 & & & & & x_1, \ldots, x_3 & \geq & 0 \\
 & & & & & x_1, x_2 & \text{integers} &
\end{array}$$

(Notice not all variables are constrained to be integers.) A Math 16 student passed away while solving the problem using the branch-and-bound algorithm. The partial tree formed is pictured below. Complete his solution.

N.B. If I were writing this question for the actual Final, there would be a tree here which would require only one pivot to complete the solution (perhaps needing justification that the optimum could not lie on another uncompleted branch). I am too lazy to do that for a practice exam, so you will have to solve it from scratch. (But please don't die in the process.)