A proof of Theorem 10.2 in Grinstead and Snell

This theorem asserts that a random variable with finitely many values is determined by its moment generating function. Here is a proof that does not involve Vandermonde determinants. Suppose X, Y are random variables with finitely many values and with the same moment generating function. We'd like to show that $X = Y$.

We somewhat "normalize" the problem in the following way: we may assume that both X and Y take the same values. Indeed, if there is some value v where $p_X(v) > 0$ and it does not look like v is a value for Y, we can always make it one, by taking $p_Y(v) = 0$, and vice versa.

Assume that $X \neq Y$. Since the sum of the probabilities for each is 1, it follows that at least two of these probabilities are unequal, and so they are unequal at some value $v \neq 0$. Next, let w be the value with $|w|$ maximal and $p_X(w) \neq p_Y(w)$, so that $w \neq 0$.

If the moment generating functions for X, Y are equal, then so too are their moments, since the moment generating functions converge everywhere, and the kth derivative at 0 is the kth moment. So assume that the moments of X, Y are equal, and so for each k, we have

$$\sum_v p_X(v) v^k = \sum_v p_Y(v) v^k.$$

For those v with $|v| > |w|$, the terms on the left side of the equation are identical to the terms on the right side of the equation, by the definition of w. Subtracting these off, we have

$$\sum_{v:\ |v| \leq |w|} p_X(v) v^k = \sum_{v:\ |v| \leq |w|} p_Y(v) v^k.$$

Now divide both sides by w^k, getting

$$\sum_{v:\ |v| \leq |w|} p_X(v) \left(\frac{v}{w}\right)^k = \sum_{v:\ |v| \leq |w|} p_Y(v) \left(\frac{v}{w}\right)^k.$$

As $k \to \infty$, the terms in these sums with $|v| < |w|$ tend to 0, so we have some function $f(k)$ that tends to 0 as $k \to \infty$ with

$$\sum_{v:\ |v| = |w|} p_X(v) \left(\frac{v}{w}\right)^k = \sum_{v:\ |v| = |w|} p_Y(v) \left(\frac{v}{w}\right)^k + f(k).$$

There are two possibilities: There is just one v with $|v| = |w|$, namely $v = w$, or there are two terms, namely $v = \pm w$. In the first case we have $v/w = 1$, so the above equation simplifies to

$$p_X(w) = p_Y(w) + f(k),$$

and letting $k \to \infty$, we see that $p_X(w) = p_Y(w)$, a contradiction. The other case is that there are the two terms $\pm w$, which gives

$$p_X(w) + (-1)^k p_X(-w) = p_Y(w) + (-1)^k p_Y(-w) + f(k).$$

Letting $k \to \infty$ through even numbers, we see that

$$p_X(w) + p_X(-w) = p_Y(w) + p_Y(-w),$$

and letting $k \to \infty$ through odd numbers, we see that

$$p_X(w) - p_X(-w) = p_Y(w) - p_Y(-w).$$

Adding these last two equations and dividing by 2, we get that $p_X(w) = p_Y(w)$, again a contradiction.

Thus, if X, Y have the same moment generating functions, they must be equal.